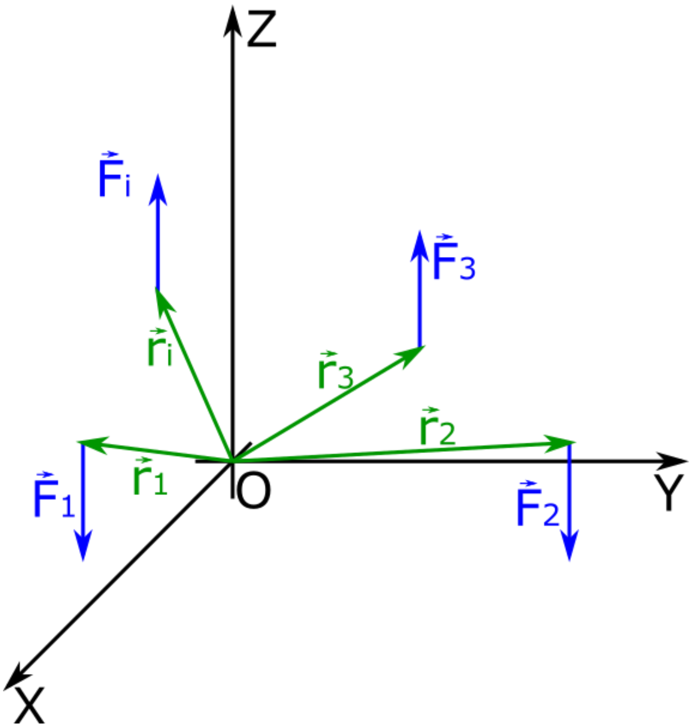
Center of gravity

To determine how we can find the center of gravity of a rigid body, we'll start by reminding you how we reduce the spatial system of parallel forces.



F_i - parallel forces

r_i - position vectors of F_i forces

We have a system of parallel forces as in the drawing above. This arrangement can be dredged to one resultant force that will pass through point C.
Point C with coordinates X_c, Y_c, Z_c is called the center of parallel forces. The location of this point does not depend on the direction of the forces of the system, but only on their value and the relative position of these forces.

$$\vec{r}_c * \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} \vec{F}_i \vec{r}_i$$

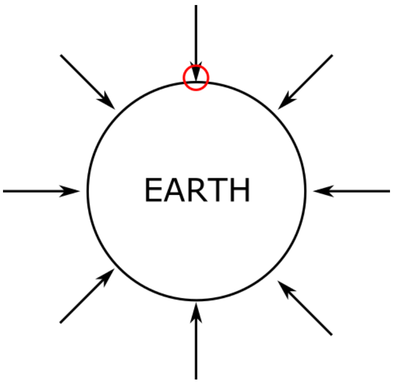
$$\vec{r}_c = \frac{\sum_{i=1}^{n} \vec{F}_i \vec{r}_i}{\sum_{i=1}^{n} \vec{F}_i}$$

Attachment point of the resultant parallel forces $\begin{cases} X_c = \frac{\sum_{i=1}^{n} \vec{F}_i x_i}{\sum_{i=1}^{n} \vec{F}_i} \\ Y_c = \frac{\sum_{i=1}^{n} \vec{F}_i y_i}{\sum_{i=1}^{n} \vec{F}_i} \\ Z_c = \frac{\sum_{i=1}^{n} \vec{F}_i \vec{z}_i}{\sum_{i=1}^{n} \vec{F}_i} \end{cases}$

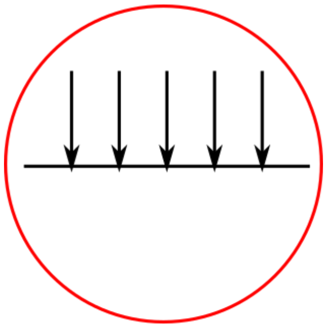
Once we know how and for what we can reduce the system of parallel forces, we can move to the center of gravity.

Why is the use of parallel forces so important for the center of gravity?

If we assume a situation in which we stand in a certain place on Earth and look at it from a distance (from space), then we can see that the forces of gravity will act then as in the drawing.



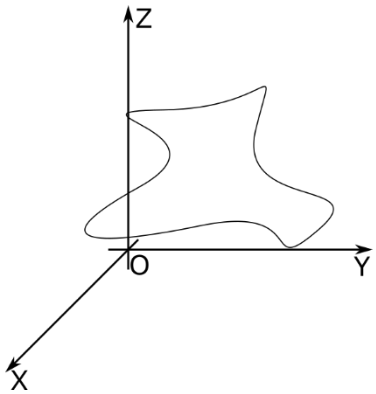
Let's approach the Earth's surface, to the point marked in red. It is clear that in such a close environment the Earth seems to be flat for us, and the gravitational forces affecting us are parallel forces. In the further part we will apply this generalization to all bodies on Earth (or in its gravitational field).



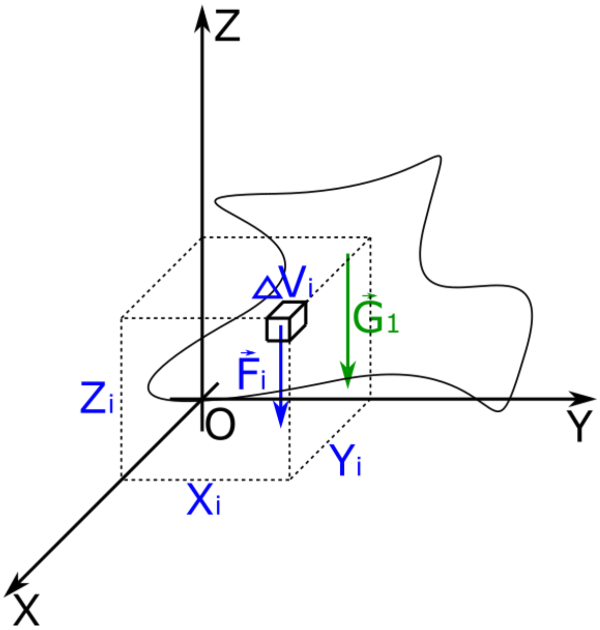
For gravity, the center of parallel forces = center of gravity

How to determine the position of the center of gravity of the body.

1. Suppose we have a body rigid in space.



2. Let's cut a tiny volume element out of such a body



ΔV_i- a tiny volume element with coordinates (x_i, y_i, z_i),

F_i – gravity force

Based on the information from the reduction of the parallel forces system shown at the beginning, we can write

$$X_c * \sum_{i=1}^{n} \vec{F_i} = \sum_{i=1}^{n} \vec{F_i} x_i$$

$$X_c = \frac{\sum_{i=1}^{n} \vec{F_i} x_i}{\sum_{i=1}^{n} \vec{F_i}}$$

Continuing, we see that in this case, we do not have explicitly recorded forces, but only the effect of these forces on volume. That's why we need to convert gravity into volume. To do this, let's introduce the specific weight γ_i of our body.
We know that gravity will be the product of specific gravity and volume.

$$\vec{F_i} = \gamma_i * \Delta V_i$$

Let us substitute the force in our equation according to the above formula.

$$X_c = \frac{\sum_{i=1}^{n} \gamma_i * \Delta V_i * x_i}{\sum_{i=1}^{n} \gamma_i * \Delta V_i}$$

We do the same with the other coordinates.

$$Y_c = \frac{\sum_{i=1}^{n} \gamma_i * \Delta V_i * y_i}{\sum_{i=1}^{n} \gamma_i * \Delta V_i}$$

$$Z_c = \frac{\sum_{i=1}^{n} \gamma_i * \Delta V_i * z_i}{\sum_{i=1}^{n} \gamma_i * \Delta V_i}$$

If we consider the case in which our body is divided into infinitely many elements, the infinitely small size we get the following equations.

$$X_c = \frac{\int \gamma * x * dV}{\int \gamma * dV} = \frac{\int \gamma * x * dV}{G}$$

$$Y_c = \frac{\int \gamma * y * dV}{\int \gamma * dV} = \frac{\int \gamma * y * dV}{G}$$

$$Z_c = \frac{\int \gamma * z * dV}{\int \gamma * dV} = \frac{\int \gamma * z * dV}{G}$$

Because we know that

$$\int \gamma * dV$$

it is the total weight of our body.

In addition, we know that the specific weight of our body is the product of body density and gravitational acceleration, so ultimately we get the following equations.

$$\gamma = \rho * g$$

$$X_c = \frac{\int \rho * g * x * dV}{G} = \frac{\int \rho * g * x * dV}{m * g} = \frac{\int \rho * x * dV}{m}$$

$$Y_c = \frac{\int \rho * g * y * dV}{G} = \frac{\int \rho * g * y * dV}{m * g} = \frac{\int \rho * y * dV}{m}$$

$$Z_c = \frac{\int \rho * g * z * dV}{G} = \frac{\int \rho * g * z * dV}{m * g} = \frac{\int \rho * z * dV}{m}$$

The above equations bind in themselves the mass of the body under consideration, hence they are called mass equations.
However, if we assume that the body in question is a homogeneous body, then the specific weight of the body will be constant, so it can be reduced in our equations.

$$X_c = \frac{\int \gamma * x * dV}{\int \gamma * dV} = \frac{\gamma \int x * dV}{\gamma \int dV} = \frac{\int x * dV}{\int dV}$$

$$Y_c = \frac{\int \gamma * y * dV}{\int \gamma * dV} = \frac{\gamma \int y * dV}{\gamma \int dV} = \frac{\int y * dV}{\int dV}$$

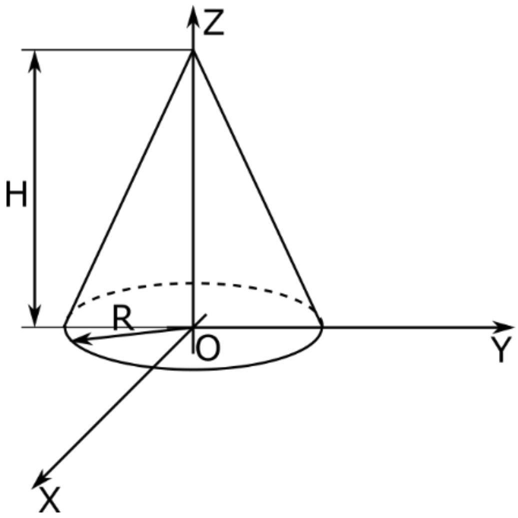
$$Z_c = \frac{\int \gamma * z * dV}{\int \gamma * dV} = \frac{\gamma \int z * dV}{\gamma \int dV} = \frac{\int z * dV}{\int dV}$$

The above equations are true for a three-dimensional body (we integrate by volume).
However, for a flat body, the equations that allow finding the center of gravity will have the following form (we integrate on the surface).

$$X_c = \frac{\int x * dA}{\int dA}$$

$$Y_c = \frac{\int y * dA}{\int dA}$$

Ex. 1. Let's try to use the above information to find the center of gravity of a given cone with dimensions H and R.

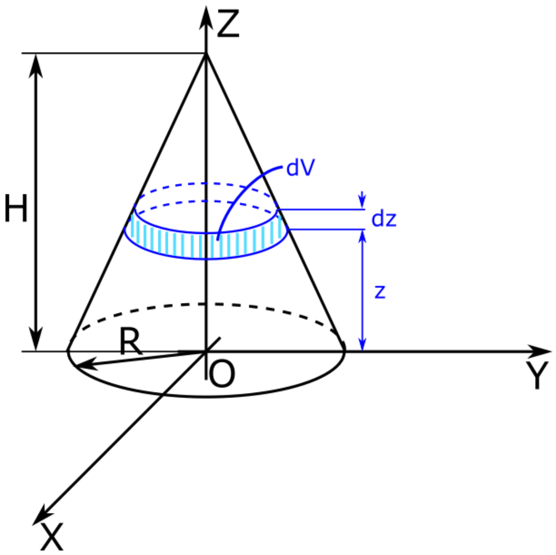


We see that the cone has one symmetry axis (Z). Therefore, we can immediately write that the coordinates of the center of gravity on the axes X and Y will be $X_c = 0$, $Y_c = 0$.

The position (coordinate) of the center of gravity on the Z axis remains to be found.
Let's use the equations we just learned.

$$Z_c = \frac{\int z * dV}{\int dV}$$

At the beginning we define what dV is for us.

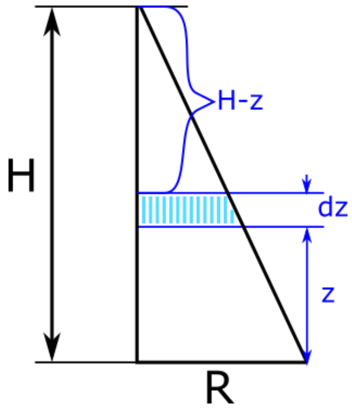


We see that our dV will be a very thin slice cut from our cone.

$$dV = \pi r^2 * dz$$

It remains for us to determine how the radius “r” will change depending on the height at which we will be.

To determine this, let's draw a section of our cone as a triangle. As shown in the picture.



looking at the above figure, we see that we get two similar triangles H, R and H-z, r. By using the similarities of triangles we get.

$$\frac{R}{H} = \frac{r}{H - z} \rightarrow r = \frac{R(H - z)}{H}$$

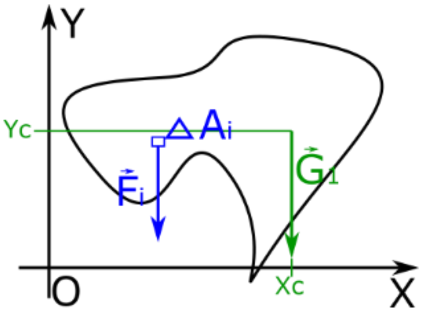
We insert the above relationship into our equation for dV and get.

$$dV = \pi \left(\frac{R(H - z)}{H}\right)^2 * dz$$

We insert the whole into our initial equation and calculate the position of the center of gravity on the Z axis.

$$Z_c = \frac{\int z * dV}{V} = \frac{\int_0^H z * dV}{V} = \frac{\int_0^H z * \pi \left(\frac{R(H - z)}{H}\right)^2 * dz}{V} \overset{\dots}{=} \frac{\frac{\pi R^2}{H^2} * \frac{1}{12} H^4}{\frac{1}{3} * \pi R^2 H} = \frac{H}{4}$$

In the case of flat figures, it is very similar as in the case of spatial objects, while instead of integrating by volume we will integrate by surface.

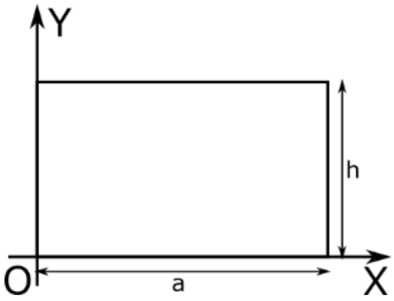


$$X_c = \frac{\int x * dA}{\int dA} = \frac{\int x * dA}{A}$$

$$Y_c = \frac{\int y * dA}{\int dA} = \frac{\int y * dA}{A}$$

For this course, we'll focus on flat figures

Ex. 2. Find the center of gravity location for the rectangle with sides a and h.

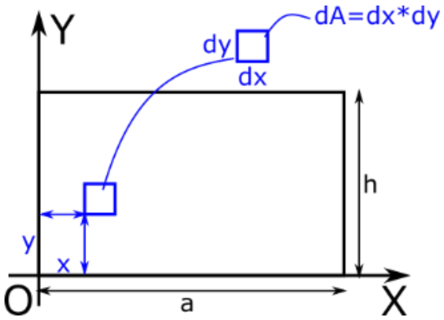


We start with our two equations

$$X_c = \frac{\int x * dA}{A}$$

$$Y_c = \frac{\int y * dA}{A}$$

Looking at our equations, we see that we will need a small volume element dA. In our case, dA = dx * dy.



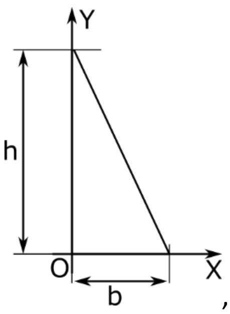
By inserting the element dA in our equations we get.

$$X_c = \frac{\int x * dA}{A} = \frac{\int_0^a \int_0^h xdxdy}{A} = \frac{\int_0^a xdx \int_0^h dy}{A} = \frac{\frac{a^2}{2} * h}{a * h} = \frac{a}{2}$$

$$Y_c = \frac{\int y * dA}{A} = \frac{\int_0^a \int_0^h ydxdy}{A} = \frac{\int_0^a dx \int_0^h ydy}{A} = \frac{\frac{h^2}{2} * a}{a * h} = \frac{h}{2}$$

It is clear that we have obtained the result to which we are accustomed, namely that the center of gravity in a homogeneous rectangle will be half its height and half its base.

Ex. 3. Find the center of gravity position for the triangle with sides a and b as shown.



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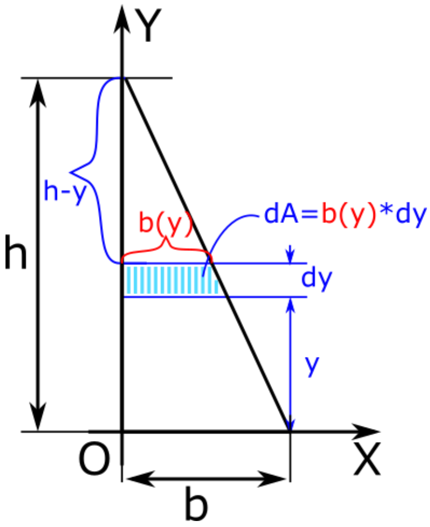
Again, we will use our basic equations.

$$X_c = \frac{\int x * dA}{A}$$

$$Y_c = \frac{\int y * dA}{A}$$

Looking at our equations, we see that we will need a small volume element dA. In the case of a triangle it will be a little more complicated. Instead of a small triangle-shaped element, as it may seem, we will cut out small rectangles, which eventually also creates a triangle for us. The problem, however, is to determine how the height of such a rectangle will change depending on where we are located.



We assume that we will first calculate the location of the center of gravity on the Y axis. In this case, dA = b(y) * dy. Where b(y) is a height of the rectangle. We need to determine how the height of our rectangle (b(y)) will change depending on Y. Again, as in the case of the cone, we will use the similarities of triangles.

$$\frac{b}{h} = \frac{b(y)}{h-y} \rightarrow b(y) = \frac{b(h-y)}{h}$$

Knowing how he changes the height depending on Y, we can proceed to solving the equation.

$$Y_c = \frac{\int y * dA}{A} = \frac{\int_0^h yb(y)dy}{A} = \frac{\int_0^h y\frac{b(h-y)}{h}dy}{A} = \frac{\int_0^h y\frac{b(h-y)}{h}dy}{\frac{bh}{2}}$$

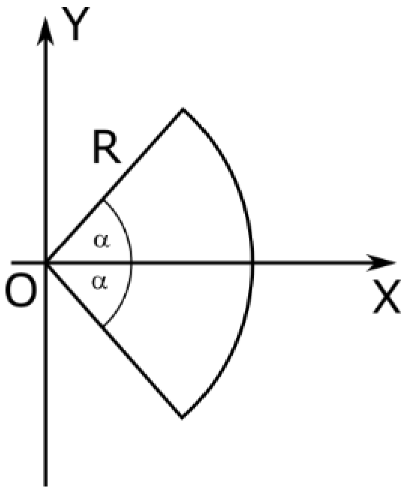
$$= \frac{\frac{b}{h}\left(h\int_0^h ydy - \int_0^h y^2\,dy\right)}{\frac{bh}{2}} \overset{\dots}{=} \frac{h}{3}$$

Now performed a similar operation in the other direction.

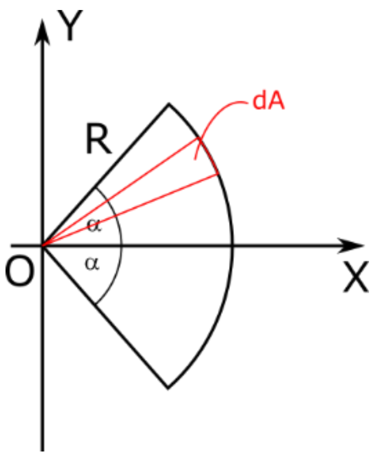
$$X_c = \frac{\int x * dA}{A} = \frac{\int_0^b xh(x)dx}{A} = \overset{\dots}{=} \frac{b}{3}$$

Finally, we received information that the center of gravity of the triangle is in one-third looking from its larger side.

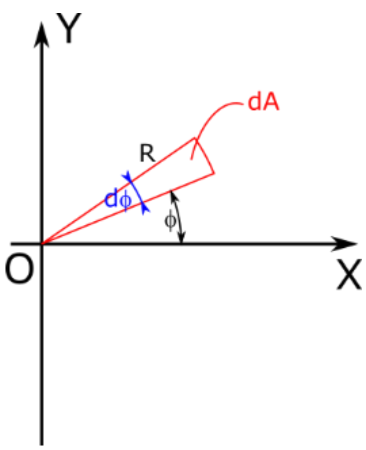
Ex. 4. Find the center of gravity position of the circle part with radius R and angle α as shown.



As in previous cases, we are dealing with a flat figure, which is why we will definitely have to specify our dA.



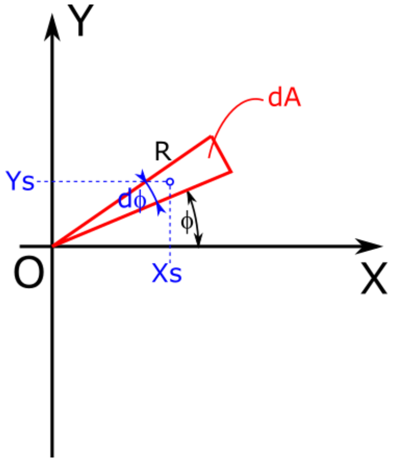
We see that dA is really just a very small circle segment.



We know that to determine the area of the circle sector, we can use the following formula.

$$dA = \frac{d\varphi}{360°} * \pi R^2 = \frac{d\varphi}{2 * \pi} * \pi R^2 = \frac{R^2}{2} d\varphi$$

While the determination of dA in the formula for the center of gravity was relatively simple, we must note that we determined it in polar coordinates, and we are looking for values for a rectangular system.



If we assume that a segment of a circle is really small, then the arc of this segment basically takes the form of a straight line and the segment itself turns into a very small triangle. And in the previous example we showed where the triangle's center of gravity is. In this case, the center of gravity of such a small rectangle will be in a third of the height when viewed from its base. Because our circle segment consists of a large number of such small triangles, the location of the center of gravity for each of these small triangles will depend on the angle as shown below.

$$X_s = \frac{2}{3} R \cos \alpha$$

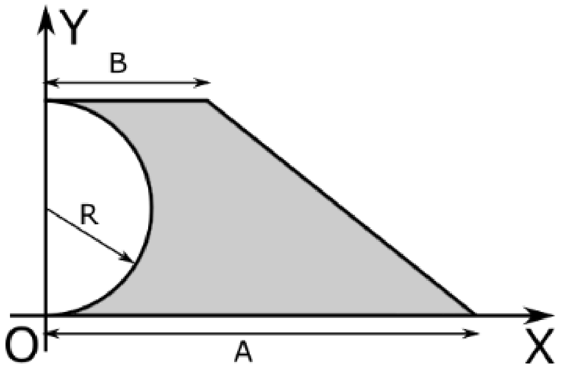
$$Y_s = \frac{2}{3} R \sin \alpha$$

The resulting equations are inserted into our basic equations, but it is clear that the X axis is the axis of symmetry of the system hence $Y_c = 0$

$$X_c = \frac{\int x * dA}{A} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3} R \cos \alpha \frac{R^2}{2} d\varphi}{\int_{-\alpha}^{\alpha} \frac{R^2}{2} d\varphi} \overset{\dots}{=} \frac{2}{3} R \frac{\sin \alpha}{\alpha}$$

$$Y_c = \frac{\int y * dA}{A} = 0$$

Ex. 5. Find the center of gravity position for the figure shown. Data: R=20, B=30, A=80.

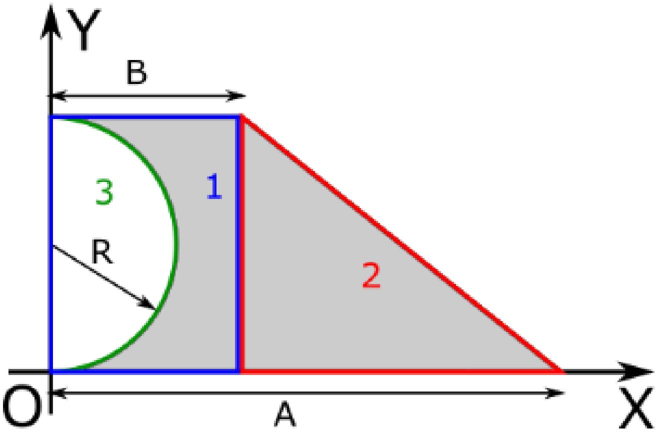


Because finding the center of gravity for such a figure as shown in the previous examples can provide a lot of problems, this example will show how to find the center of gravity for more complex shapes.

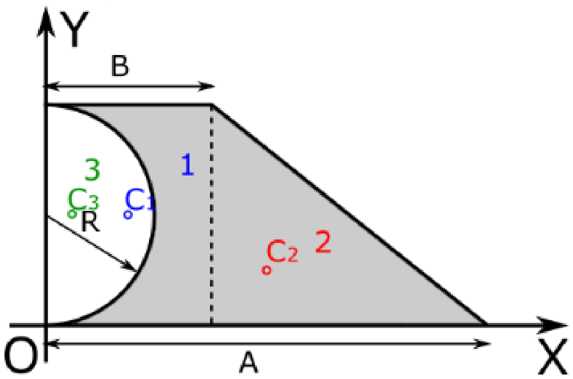
1. First, the previous examples, which were calculated from the center of gravity definition, will be needed to be able to calculate the above example.
2. Let's come back to our definition again for a moment. We see that the denominator of our equation is the surface area. However, the meter has the product of the sum of values along the appropriate axis, i.e. the coordinate of the center of gravity and the surface area in general.

$$X_c = \frac{\int x * dA}{A}$$

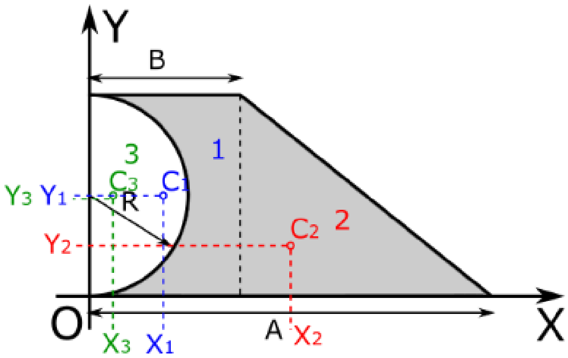
3. Based on the above examples, we can conclude that we know where the centers of gravity of simple figures are. In order to be able to find the center of gravity in our case, let's first divide our figure into simple figures as in the drawing.



4. We can see that we can divide our figure into 3 simple rectangle, triangle and half circle figures. In the next step, for each of these figures, we can figure the location of its center of gravity.



5. Next, we describe the location of each center of gravity using coordinates



$$X_1 = 15 \qquad Y_1 = 20$$

$$X_2 = 30 + \frac{1}{3} * 50 = 46{,}7 \qquad Y_2 = \frac{1}{3} * 40 = 13{,}33$$

$$X_3 = \frac{2}{3\alpha} R sin\alpha = \frac{2}{3\frac{\pi}{2}} R sin\frac{\pi}{2} = 8{,}5 \qquad Y_3 = 20$$

6. Once we know the position of the center of gravity of each figure, we will still need the area of each designated figure.

$$A_1 = 30 * 40 = 1200$$
$$A_2 = 0{,}5 * 50 * 40 = 1000$$
$$A_3 = 0{,}5 * \pi * 20^2 = 628$$

7. Once we know the surface areas and coordinates of the centers of gravity of each of our figures, we can modify our initial equation.

$$X_c = \frac{\int x * dA}{A} = \frac{A_1 * X_1 + A_2 * X_2 - A_3 * X_3}{A_1 + A_2 - A_3}$$

$$Y_c = \frac{\int y * dA}{A} = \frac{A_1 * Y_1 + A_2 * Y_2 - A_3 * Y_3}{A_1 + A_2 - A_3}$$

Why equations have the following form. Because if we add a rectangle (1) to the triangle (2) then we have to subtract half the circle (3) from whole to get the original figure.

8. We finally receive

$$X_c = \frac{A_1 * X_1 + A_2 * X_2 - A_3 * X_3}{A_1 + A_2 - A_3} = 37{,}8$$

$$Y_c = \frac{A_1 * Y_1 + A_2 * Y_2 - A_3 * Y_3}{A_1 + A_2 - A_3} = 15{,}73$$