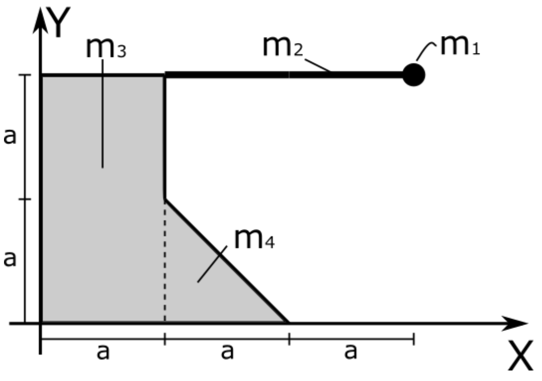
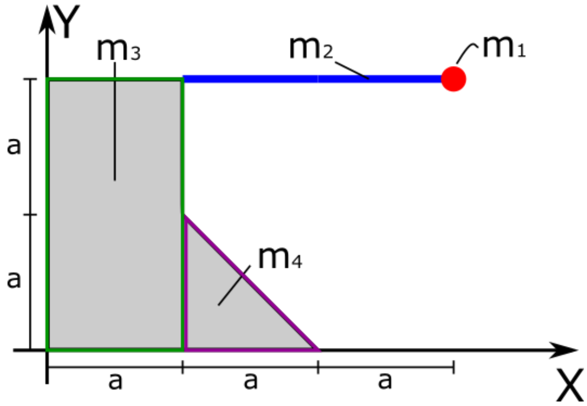
Inertia moments

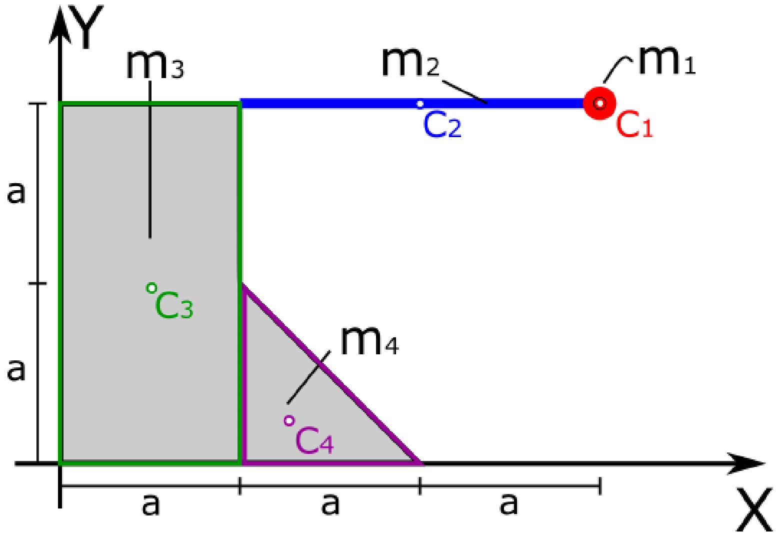
Ex. 5. For the shape shown in the figure, find the main central moments of inertia. Data: m_1, $m_2=m_1$, $m_3=2m_1$, $m_4=3m_1$.



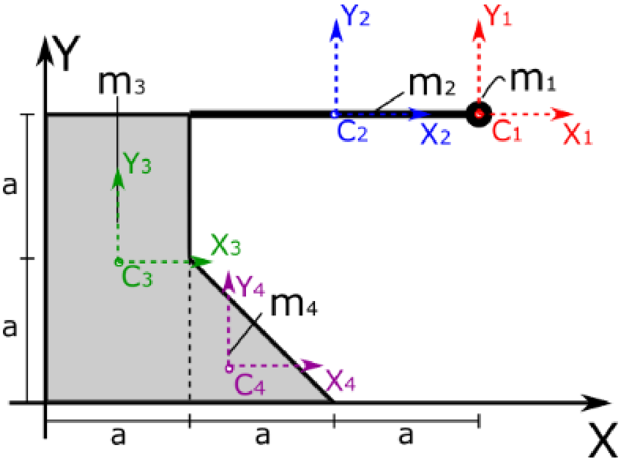
1. In this example, basically you can see the division of the shape into simple elements. This was done due to the fact that the example will be shown for mass moments. However, if the problem concerns only the geometric shape (assuming that the density is constant and is 1), then everyone will have to make such a division by themselves.



2. We divide the figure into four simple shapes: 1 - material point, 2 - rod, 3 - rectangle and 4 - triangle.



3. For such figures, we can determine the location of the center of gravity for each of the elements into which we divided our shape.



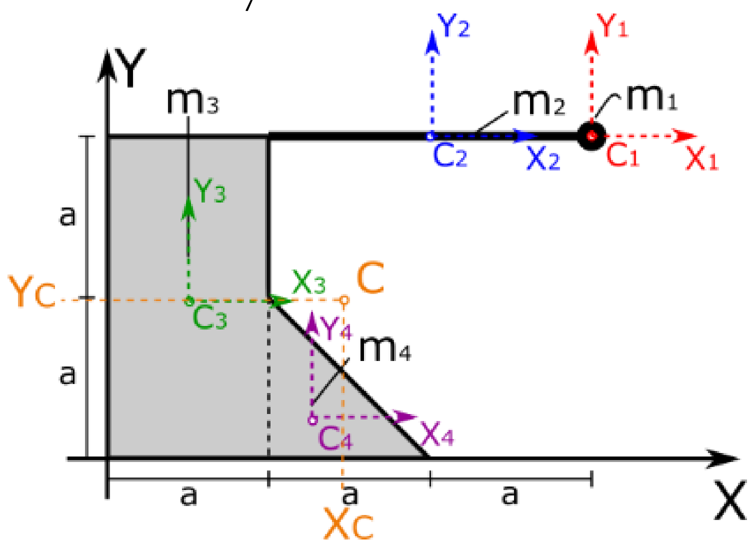
4. In the next step, we must determine the location of the center of gravity of each simple element using coordinates in the adopted system.

$$X_1 = 3a; \quad Y_1 = 2a;$$
$$X_2 = 2a; \quad Y_2 = 2a;$$
$$X_3 = 0{,}5a; \quad Y_3 = a;$$
$$X_4 = \frac{4}{3}a; \quad Y_4 = \frac{1}{3}a;$$

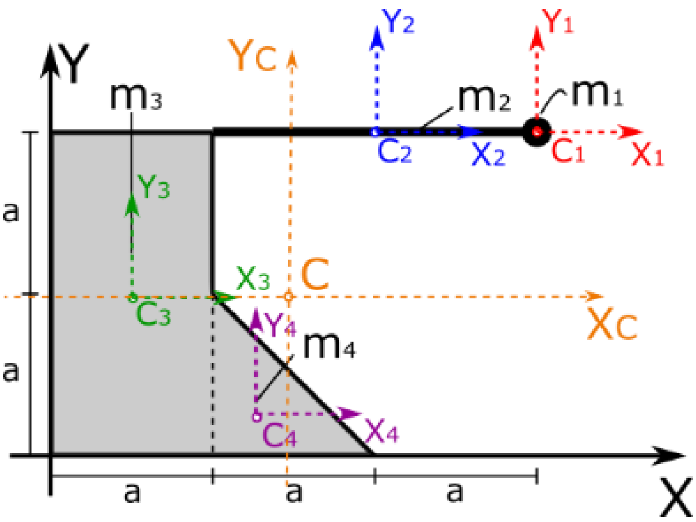
5. Once we have specified coordinates for each center of gravity, we can determine the center of gravity for the whole figure.

$$X_c = \frac{X_1 * m_1 + X_2 * m_2 + X_3 * m_3 + X_4 * m_4}{m_1 + m_2 + m_3 + m_4}$$
$$= \frac{3a * m_1 + 2a * m_1 + 0{,}5a * 2m_1 + \frac{4}{3}a * 3m_1}{m_1 + m_1 + 2m_1 + 3m_1} = \frac{10}{7}a$$

$$Y_c = \frac{Y_1 * m_1 + Y_2 * m_2 + Y_3 * m_3 + Y_4 * m_4}{m_1 + m_2 + m_3 + m_4} = \frac{2a * m_1 + 2a * m_1 + a * 2m_1 + \frac{1}{3}a * 3m_1}{m_1 + m_1 + 2m_1 + 3m_1}$$
$$= \frac{7}{7}a = a$$



6. In this way, we determined the location of the center of gravity of the whole figure, and more axes passing through the center of gravity of the system. We will need these axes to determine the main central moments of inertia.



7. We will need these axes to determine the main central moments of inertia.
8. In the next step, for each of the separated shapes, moments of inertia and moments of deviation relative to the axes passing through their centers of gravity.

$$I_{x1} = 0; \quad I_{y1} = 0; \quad D_{x1y1} = 0;$$

$$I_{x2} = 0; \quad I_{y2} = \frac{m_2(2a)^2}{12}; \quad D_{x2y2} = 0;$$

$$I_{x3} = \frac{m_3(2a)^2}{12}; \quad I_{y3} = \frac{m_3a^2}{12}; \quad D_{x3y3} = 0;$$

$$I_{x4} = \frac{m_4a^2}{18}; \quad I_{y4} = \frac{m_4a^2}{18}; \quad D_{x4y4} = -\frac{m_4aa}{36};$$

Once we know the values of the moments of inertia, relative to the axes passing through the center of gravity of each figure, we can proceed to the next step. In the next step, we will create a parallel transformation to check how much the moment of inertia will be for each figure relative to the coordinate system passing through the center of gravity of the entire shape. The solution will be shown in detail for one of the axes, the other axis will be similar. We are interested in Steiner's theorem which he says

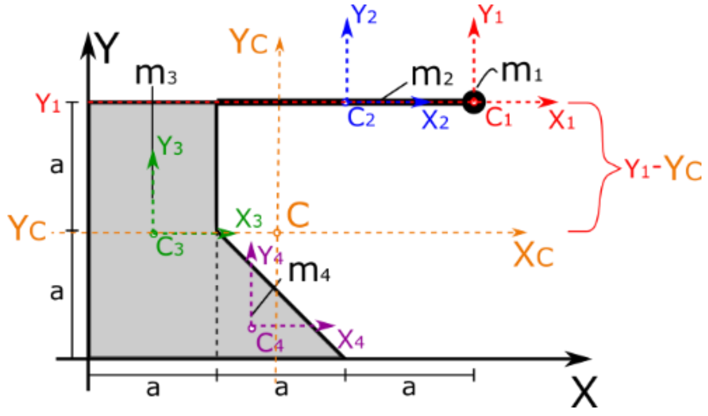
$$I_O = I_C + mr_C^2$$

Because we are interested in shifting from systems passing through the centers of gravity of simple figures to the center of gravity of the entire shape, we must write the following equation.

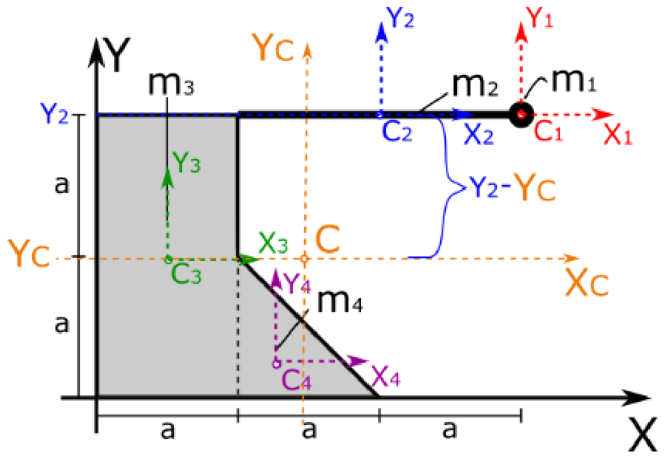
$$I_{xc} = I_{x1} + m_1{r_1}^2 + I_{x2} + m_2{r_2}^2 + I_{x3} + m_3{r_3}^2 + I_{x4} + m_4{r_4}^2$$

Because we want to know the moment of inertia about the central axes for the whole figure, we put the sum of Steiner's theorem for each of the simple figures in the equation.

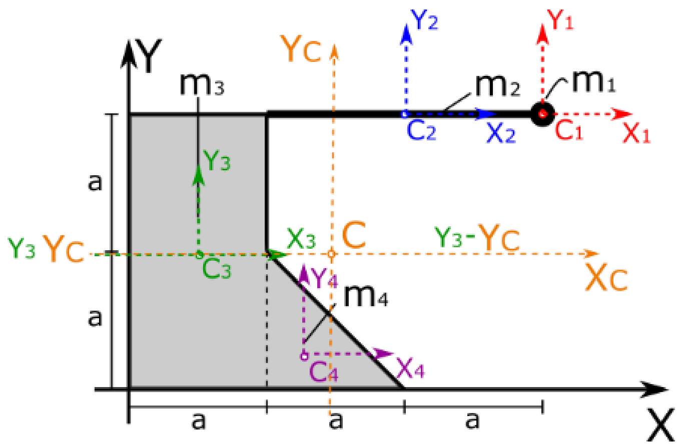
The distances marked in the equation as r with the appropriate index are actually differences in the coordinates of the respective axes. When calculating moments of inertia relative to the X axis, each of the X axes passes through the coordinate on the Y axis. To better understand this, please look at the following images together with information on how to write r for each of the individual figures



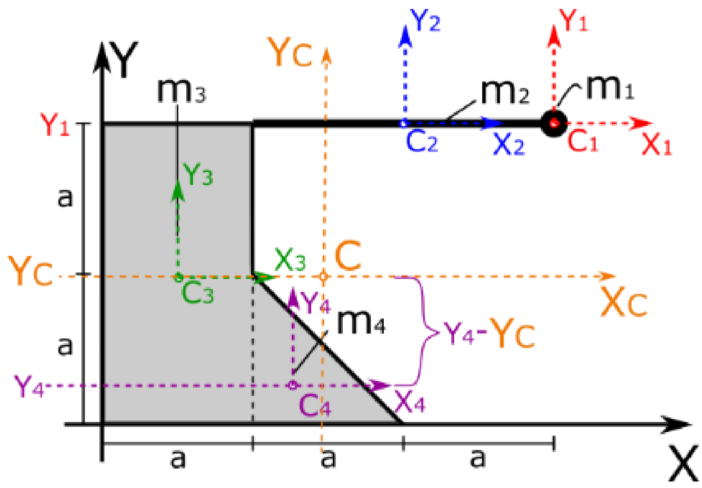
$$r_1 = (y_1 - y_c)$$



$$r_2 = (y_2 - y_c)$$



$$r_3 = (y_3 - y_c)$$



$$r_4 = (y_4 - y_c)$$

9. Ultimately, our equation takes the following form for the X axis

$$I_{xc} = I_{x1} + m_1(y_1 - y_c)^2 + I_{x2} + m_2(y_2 - y_c)^2 + I_{x3} + m_3(y_3 - y_c)^2 + I_{x4} + m_4(y_4 - y_c)^2$$

for the Y axis

$$I_{yc} = I_{y1} + m_1(x_1 - x_c)^2 + I_{y2} + m_2(x_2 - x_c)^2 + I_{y3} + m_3(x_3 - x_c)^2 + I_{y4} + m_4(x_4 - x_c)^2$$

for the moment of deviation

$$D_{xcyc} = D_{x1y1} + m_1(x_1 - x_c)(y_1 - y_c) + D_{x2y2} + m_2(x_2 - x_c)(y_2 - y_c) + D_{x3y3} + m_3(x_3 - x_c)(y_3 - y_c) + D_{x4y4} + m_4(x_4 - x_c)(y_4 - y_c)$$

10. In the next step, we insert values into our equations

$$I_{xc} = I_{x1} + m_1(y_1 - y_c)^2 + I_{x2} + m_2(y_2 - y_c)^2 + I_{x3} + m_3(y_3 - y_c)^2 + I_{x4} + m_4(y_4 - y_c)^2$$
$$= 0 + m_1(2a - a)^2 + 0 + m_1(2a - a)^2 + \frac{2m_1(2a)^2}{12} + 2m_1(a - a)^2 + \frac{3m_1a^2}{18} + 3m_1\left(\frac{1}{3}a - a\right)^2 = \frac{25}{6}m_1a^2$$

$$I_{yc} = I_{y1} + m_1(x_1 - x_c)^2 + I_{y2} + m_2(x_2 - x_c)^2 + I_{y3} + m_3(x_3 - x_c)^2 + I_{y4} + m_4(x_4 - x_c)^2$$
$$= 0 + m_1\left(3a - \frac{10}{7}a\right)^2 + \frac{m_1(2a)^2}{12} + m_1\left(2a - \frac{10}{7}a\right)^2 + \frac{2m_1a^2}{12} + 2m_1\left(\frac{1}{2}a - \frac{10}{7}a\right)^2 + \frac{3m_1a^2}{18} + 3m_1\left(\frac{4}{3}a - \frac{10}{7}a\right)^2 = 5\frac{3}{14}m_1a^2$$

$$D_{xcyc} = D_{x1y1} + m_1(x_1 - x_c)(y_1 - y_c) + D_{x2y2} + m_2(x_2 - x_c)(y_2 - y_c) + D_{x3y3} + m_3(x_3 - x_c)(y_3 - y_c) + D_{x4y4} + m_4(x_4 - x_c)(y_4 - y_c)$$
$$= 0 + m_1\left(3a - \frac{10}{7}a\right)(2a - a) + 0 + m_1\left(2a - \frac{10}{7}a\right)(2a - a) + 0 + 2m_1\left(\frac{1}{2}a - \frac{10}{7}a\right)(a - a) - \frac{m_4 aa}{36} + 3m_1\left(\frac{4}{3}a - \frac{10}{7}a\right)\left(\frac{1}{3}a - a\right)$$
$$= 2\frac{1}{4}m_1a^2$$

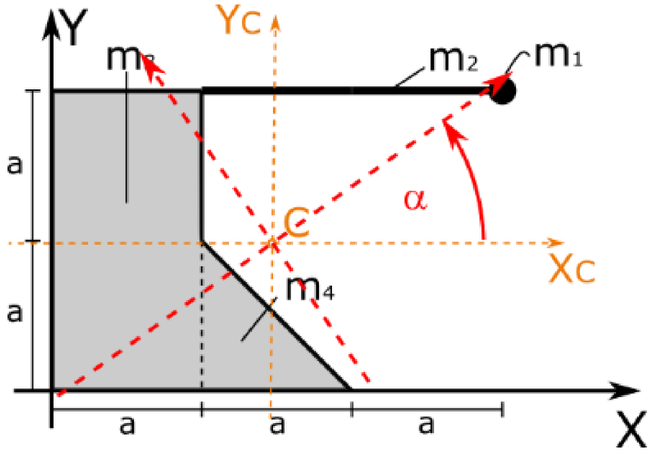
11. When we have counted the central moments of inertia, we can proceed to the calculation of the main central moments of inertia. For this purpose, we will use the equations given for rotational transformation of inertia moments.

$$\alpha_O = \frac{1}{2}arctan\left(\frac{2D_{xy}}{I_y - I_x}\right)$$
$$I_1 = I_{max} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2}$$
$$I_2 = I_{min} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2}$$

12. First, we check the angle by which the central coordinate system should be rotated to find the main axis system.

$$\alpha_O = \frac{1}{2}arctan\left(\frac{2D_{xy}}{I_y - I_x}\right) = \frac{1}{2}arctan\left(\frac{2D_{xcyc}}{I_{yc} - I_{xc}}\right) = \frac{1}{2}arctan\left(\frac{2 * 2\frac{1}{4}m_1a^2}{5\frac{3}{14}m_1a^2 - \frac{25}{6}m_1a^2}\right)$$
$$= 38°26'$$

13. We rotate the central system by the calculated angle



14. We calculate the values of the moments of inertia relative to the rotated coordinate system

$$I_1 = I_{max} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2}$$
$$= \frac{1}{2}(I_{xc} + I_{yc}) + \frac{1}{2}\sqrt{(I_{xc} - I_{yc})^2 + 4D_{xcyc}{}^2}$$
$$= \frac{1}{2}\left(\frac{25}{6}m_1a^2 + 5\frac{3}{14}m_1a^2\right)$$
$$+ \frac{1}{2}\sqrt{\left(\frac{25}{6}m_1a^2 - 5\frac{3}{14}m_1a^2\right)^2 + 4 * \left(2\frac{1}{4}m_1a^2\right)^2} = 7m_1a^2$$

$$I_2 = I_{min} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2}$$

$$= \frac{1}{2}(I_{xc} + I_{yc}) - \frac{1}{2}\sqrt{(I_{xc} - I_{yc})^2 + 4D_{xcyc}{}^2}$$

$$= \frac{1}{2}\left(\frac{25}{6}m_1a^2 + 5\frac{3}{14}m_1a^2\right)$$

$$- \frac{1}{2}\sqrt{\left(\frac{25}{6}m_1a^2 - 5\frac{3}{14}m_1a^2\right)^2 + 4 * \left(2\frac{1}{4}m_1a^2\right)^2} = 2{,}38m_1a^2$$

15. Once we know the values of the main central moments of inertia, it remains to mark them on the axes. For this purpose, we check what sign was at the calculated moment of deviation relative to the central axes. We see that the value of the moment of deviation is greater than zero, therefore the axes will be marked as shown.