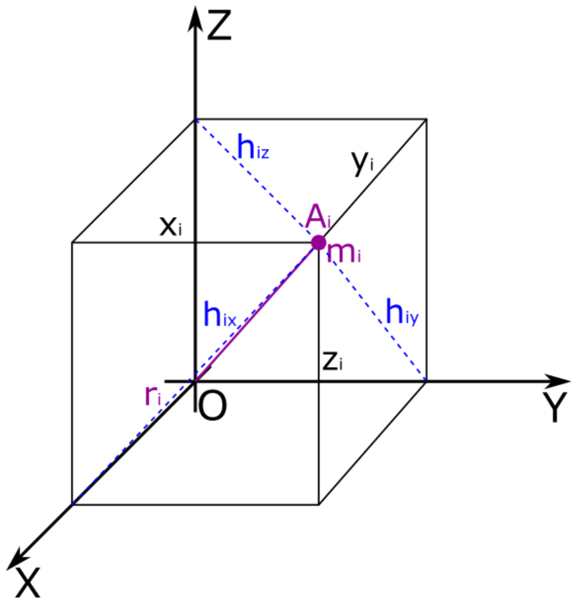
Inertia moments

We will begin this subject with moments of inertia for the system of material points.



Let us assume that we have a material system consisting of n material points with masses m_i, located at points A_i described by the leading vectors r_i.

$$\vec{r_i} = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$$

<u>The polar moment of inertia I_o</u> of the system of material points relative to point O is the sum of the products of masses m_i and squares of their distance r_i^2 from point O, i.e.

$$I_O = \sum_{i=1}^{n} m_i\, r_i^2 = \sum_{i=1}^{n} m_i\,(x_i^2 + y_i^2 + z_i^2)$$

<u>The moments of inertia I_{xy}, I_{yz}, I_{zx} with respect to the xy, yx, zx planes</u> of the material points system are the sums of products of masses m_i and squares of their distance from these planes. So we have:

$$I_{xy} = \sum_{i=1}^{n} m_i\, z_i^2; \quad I_{yz} = \sum_{i=1}^{n} m_i\, x_i^2; \quad I_{zx} = \sum_{i=1}^{n} m_i\, y_i^2;$$

<u>The moments of inertia I_x, I_y, I_z relative to the x, y, z axis</u>, from the system of material points are the sums of products of masses m_i and squares of their distance from these axes. So we have:

$$I_x = \sum_{i=1}^{n} m_i\, h_{ix}^2 = \sum_{i=1}^{n} m_i\,(y_i^2 + z_i^2)$$
$$I_y = \sum_{i=1}^{n} m_i\, h_{iy}^2 = \sum_{i=1}^{n} m_i\,(x_i^2 + z_i^2)$$
$$I_z = \sum_{i=1}^{n} m_i\, h_{iz}^2 = \sum_{i=1}^{n} m_i\,(x_i^2 + y_i^2)$$

In addition to the moments of inertia defined above with respect to the point, planes and axes, also the quantities that we call deviant moments play an important role.

The deviant moments D_{xy}, D_{yz}, D_{zx} of the system of material points are the sum of the products of masses mi with products of the distance from two perpendicular planes, yz and zx, zy and xy, xy and yz. These moments are expressed by the formulas:

$$D_{xy} = D_{yx} = \sum_{i=1}^{n} m_i \, x_i y_i$$

$$D_{yz} = D_{zy} = \sum_{i=1}^{n} m_i \, y_i z_i$$

$$D_{xz} = D_{zx} = \sum_{i=1}^{n} m_i \, z_i x_i$$

Moments of deviation can have both positive and negative values, because in the above formulas, as opposed to moments of inertia, there are products of coordinates, not coordinate squares.

In addition, if one of the two planes with respect to which we calculate deviant moments is the plane of symmetry of the material system, then the relevant deviant moments will be zero.

Suppose the plane of symmetry is the xy plane. In this case, for each point A_i with the coordinates x_i, y_i, z_i and mass m_i correspond to the principle of symmetry, another point Ai' with the coordinates x_i, y_i, $-z_i$ with the same mass m_i. The deviation moments of two points will be zero.

$$m_i x_i z_i + m_i x_i(-z_i) = m_i x_i(z_i - z_i) = 0$$

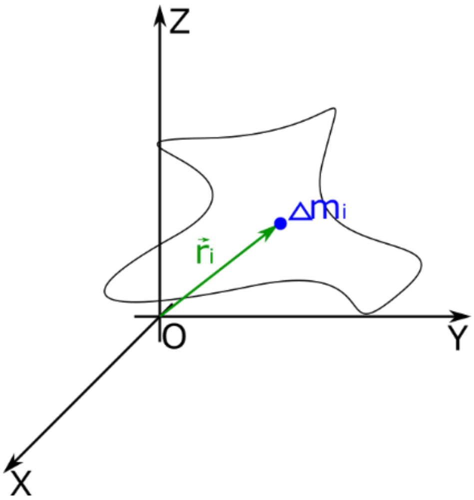
$$m_i y_i z_i + m_i y_i(-z_i) = m_i y_i(z_i - z_i) = 0$$

that is, two of the three moments of deviation will be zero

$$D_{zx} = D_{yz} = 0$$

It is easy to notice that if the material system has two planes of symmetry, then all deviation moments will be equal to zero.

Moments of inertia of the solid



If we divide a solid with mass m into n small elements with masses Δm_i, then approximate values of moments of inertia of these elements, treated as material points, can be calculated from formulas for moments for the system of material points.

The exact values of the moments of inertia are obtained by taking the sum limit with the number of elements n striving to infinity and the mass striving to zero. Then instead of sums we get the whole mass m.

Polar moment of inertia

$$I_O = \lim_{n\to\infty} \sum_{i=1}^{n} m_i r_i^2 = \int r^2\, dm = \int (x^2 + y^2 + z^2) dm$$

It is known from the integral calculus that the above equation can be broken down into the sum of individual integrals.

$$I_O = \int (x^2 + y^2 + z^2) dm = \int x^2\, dm + \int y^2 dm + \int z^2 dm$$

Integrals occurring in the above expression are **moments of inertia relative to the planes**.

$$I_{xy} = \int z^2 dm\, ; \quad I_{yz} = \int x^2\, dm; \quad I_{zx} = \int y^2 dm$$

The formula for the polar moment of inertia results in a relationship

$$I_O = I_{xy} + I_{yz} + I_{zx}$$

The polar moment of inertia is equal to the sum of the moments of inertia relative to the three planes passing through this pole.

Dependencies on **moments of inertia relative to the axis** are:

$$I_x = \int (y^2 + z^2)\, dm = \int y^2 dm + \int z^2 dm$$
$$I_y = \int (z^2 + x^2)\, dm = \int z^2 dm + \int x^2\, dm$$
$$I_z = \int (x^2 + y^2)\, dm = \int x^2\, dm + \int y^2 dm$$

From the above formulas, the relationship between the moments of inertia about the axis and about the planes can be seen.

The moment of inertia about the axis is equal to the sum of the moments of inertia about two planes intersecting along this axis.

By adding formulas and taking into account the relationship to the moment of inertia with respect to the pole, we get

$$I_O = \frac{1}{2}(I_x + I_y + I_z)$$

The polar moment of inertia is equal to half the sum of the moments of inertia about the three perpendicular axes passing through that pole.

Deviant moments for a solid can be written as

$$D_{xy} = D_{yx} = \int xydm$$
$$D_{yz} = D_{zy} = \int yzdm$$
$$D_{xz} = D_{zx} = \int xzdm$$

If we substitute the dependence for all previous equations that dm = ρdV, where ρ - the density of the solid at the point with coordinates x, y, z, and V volume and we assume that the solid is homogeneous, then we will obtain formulas in the following form.

Polar moment of inertia

$$I_O = \rho \int (x^2 + y^2 + z^2) dV$$

Moments of inertia relative to the planes

$$I_{xy} = \rho \int z^2 dV\,; \quad I_{yz} = \rho \int x^2\, dV; \quad I_{zx} = \rho \int y^2 dV$$

Moments of inertia relative to the axis

$$I_x = \rho \int (y^2 + z^2)\, dV$$
$$I_y = \rho \int (z^2 + x^2)\, dV$$
$$I_z = \rho \int (x^2 + y^2)\, dV$$

Deviant moments for a solid

$$D_{xy} = D_{yx} = \rho \int xydV$$
$$D_{yz} = D_{zy} = \rho \int yzdV$$
$$D_{xz} = D_{zx} = \rho \int xzdV$$

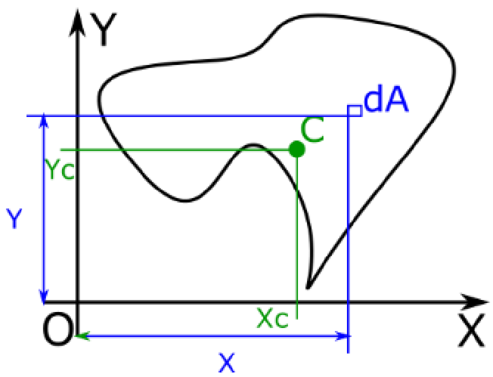
The integrals found in the above formulas are called geometric moments of inertia, which depend only on the shape of the body.

Each moment of inertia I can be generally written as the product of the total mass of the system m and the square of a certain distance r^2 from the adopted plane, axis or pole. This distance is called the radius of inertia of the body relative to a given plane, axis, pole.

$$I = mr^2$$

The moment of inertia defined in this way has practical application when calculating the moments of inertia of machine elements.

Moments of inertia and deviation on a plane



The moment of inertia of given figure relative to the O_x axis will be called

$$I_x = \rho \int y^2 \, dA$$

he moment of inertia of given figure relative to the O_y axis will be called

$$I_y = \rho \int x^2 \, dA$$

Knowing the values of the moments of inertia, you can determine the so-called arm or radius of inertia

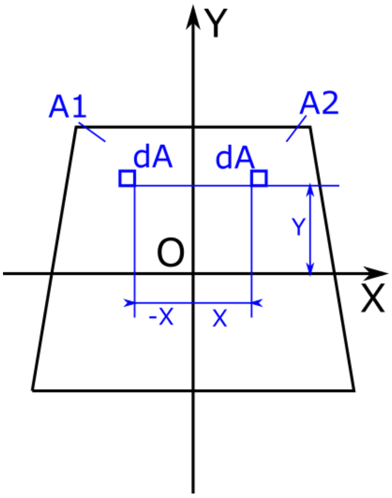
$$i_x = \sqrt{\frac{I_x}{A}}; \quad i_y = \sqrt{\frac{I_y}{A}};$$

A moment of deviation

$$D_{xy} = \rho \int xydA$$

The x and y coordinates indicate the coordinates of the center of gravity of the surface area element dA in the given O_x and O_y axis system.

Similarly to the 3D system in a planar system, the deviation moment will be zero if one of the axes is the axis of symmetry.



In the drawing, we have a figure with one axis of symmetry O_y. Let's divide the field of figures into two symmetrical elements with respect to this axis. Each element of the field with coordinates x, y corresponds to a symmetrical element with coordinates -x, y. The moments of deviation of such two elements with respect to the axes O_x and O_y are equal in absolute value and differ only by a sign.

$$Dxy = \int (-x) y dA_1 + \int xy dA_2 = 0$$

The polar moment of inertia can be defined as

$$I_O = \int r^2 \, dA$$

we know that

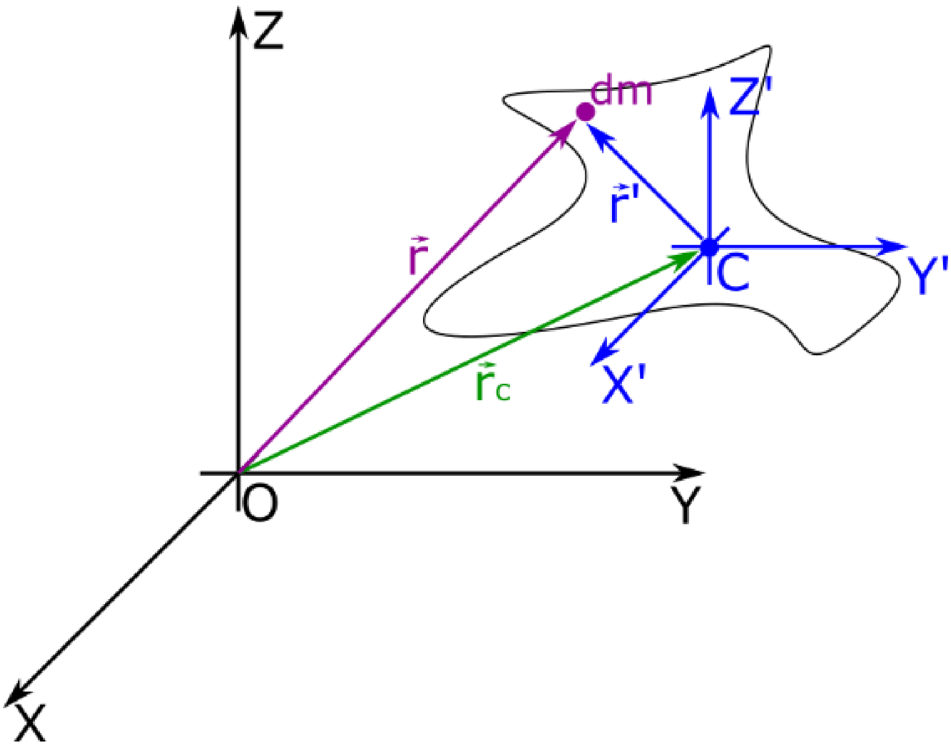
$$r^2 = x^2 + y^2$$

hence

$$I_O = \int r^2 \, dA = \int x^2 + y^2 \, dA = \int x^2 \, dA + \int y^2 \, dA = I_x + I_y$$

The polar moment of inertia is equal to the sum of the axial moments of inertia.

Parallel transformation of moments of inertia (Steiner's theorem)



Let's assume two coordinate systems x, y, z and x ', y', z' with parallel axes respectively. The system x, y, z originates at any point O, and the system x ', y', z' in the center of mass of the solid C. The center of mass of the solid in the x, y, z system is described by the r_c vector.

$$\vec{r_c} = x_c\hat{\imath} + y_c\hat{\jmath} + z_c\hat{k}$$

The position of the mass element dm is determined in the x, y, z system by the vector r.

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

and in the system x ', y', z' through the vector r'.

$$\vec{r'} = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k}$$

these vectors are related by dependence

$$\vec{r} = \vec{r_c} + \vec{r'}$$

therefore the coordinates of the mass element in the x, y, z system will express formulas

$$x = x_c + x'; \quad y = y_c + y'; \quad z = z_c + z';$$

The polar moment of inertia relative to point O is expressed by the formula

$$I_O = \int r^2\,dm = \int (r_c + r')^2\,dm = \int {r_c}^2\,dm + 2\int r_c r' dm + \int {r'}^2\,dm$$
$$= {r_c}^2 \int dm + 2r_c \int r'\,dm + \int {r'}^2\,dm$$

The first integral is the total mass of the solid, and the second is the static moment relative to the center of mass, i.e. it is equal to zero. Therefore

$$m = \int dm$$

and

$$\int r' \, dm = 0$$

The third of integrals is the polar moment of inertia relative to the center of mass

$$I_c = \int r'^2 \, dm$$

Ultimately the polar moment of inertia relative to any point

$$I_O = I_c + \ m{r_c}^2$$

The above theorem is called the Steiner theorem or parallel transformation of moments of inertia. The moment of inertia of a material body relative to any axis is equal to the sum of the moment of inertia relative to the parallel axis passing through the center of mass and the product of the mass and the square of the distance between the axes.

Steiner's theorem on planes

$$\begin{gathered} I_{xy} = I_{x\prime y\prime} + m{z_c}^2 \\ I_{yz} = I_{y\prime z\prime} + m{x_c}^2 \\ I_{zx} = I_{z\prime x\prime} + m{y_c}^2 \end{gathered}$$

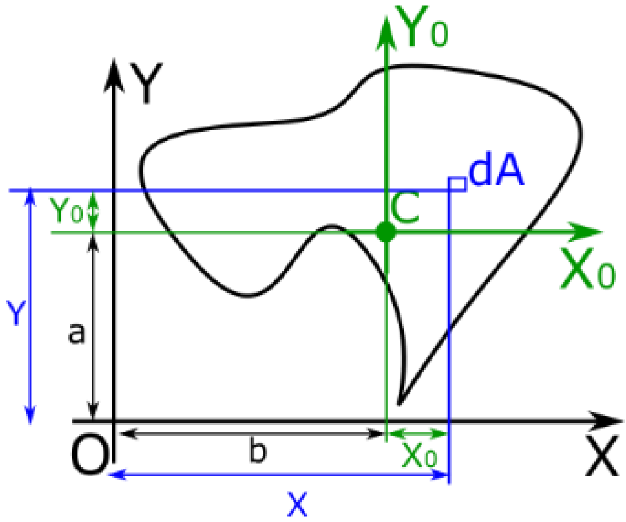
Steiner's theorem on the axis

$$\begin{gathered} I_x = I_{x\prime y\prime} + I_{z\prime x\prime} + m({y_c}^2 + {z_c}^2) \\ I_y = I_{x\prime y\prime} + I_{y\prime z\prime} + m({z_c}^2 + {x_c}^2) \\ I_z = I_{y\prime z\prime} + I_{z\prime x\prime} + m({x_c}^2 + {y_c}^2) \end{gathered}$$

Steiner theorem for moments of deviation

$$\begin{gathered} D_{xy} = D_{x\prime y\prime} + m x_c y_c \\ D_{yz} = D_{y\prime z\prime} + m y_c z_c \\ D_{zx} = D_{z\prime x\prime} + m z_c x_c \end{gathered}$$

Steiner theorem for a plane system



The coordinates of point C (center of gravity) in the Oxy system have the values a and b.

$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

By entering relationships between coordinates of a point with a parallel axis shift

$$x = x_O + b; \quad y = y_O + a;$$

$$I_x = \int (y_O + a)^2 \, dA = \int {y_O}^2 \, dA + 2 \int y_O \, a dA + \int a^2 \, dA$$
$$= {y_O}^2 \int dA + 2 y_O \int a \, dA + \int a^2 \, dA$$

After the reduction, similarly as in the case of a solid, we get the Steiner's theorem for moments of inertia in the case of a plane system.

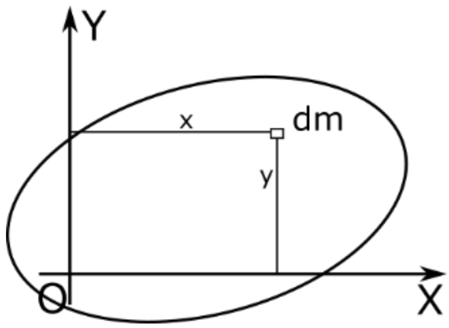
$$I_X = I_{XO} + \ a^2 A$$

$$I_Y = I_{YO} + \ b^2 A$$

And for moments of deviation

$$D_{XY} = D_{XOYO} + \ abA$$

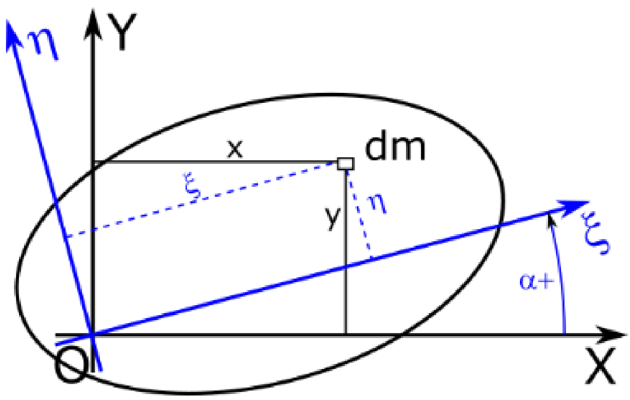
Rotational transformation of moments of inertia



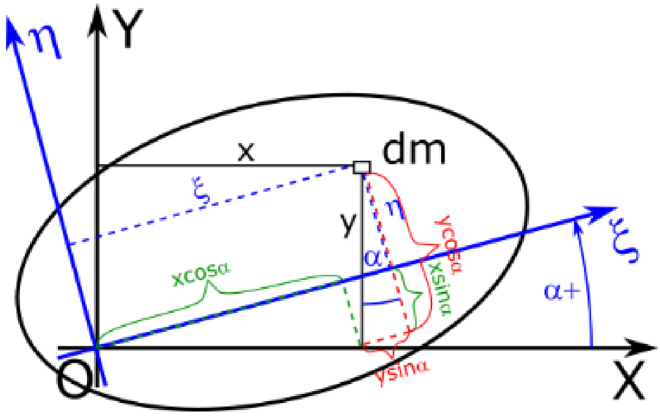
To be able to talk about rotational transformation, the definition of the main axes of inertia should be clarified at the beginning.

The main axes of inertia - the axes of the coordinate system, perpendicular to each other having the property that the moments of deviation relative to these axes are zero. If these axes pass through the center of mass of the system (point C), we call them the main central inertia axes, one of which is relatively maximum and the other minimum.

In the drawing above, it can be clearly seen that the figure is not in the major axis system. Let's introduce a system that will be initially rotated by a certain angle α.



Now we need to find the moments of inertia in the rotated coordinate system. To do this, save the coordinates of the dm element in the η, ξ rotated system.



$$\xi = y\sin\alpha + x\cos\alpha$$

$$\eta = y\cos\alpha - x\sin\alpha$$

Having the coordinates of the element dm in the rotated system, we can calculate the moments of inertia for this system I_ξ, I_η relative to the axis O_η, O_ξ and the moment of deviation $D_{\xi\eta}$.

$$I_\xi = \int \eta^2\, dm = \int (y\cos\alpha - x\sin\alpha)^2 dm = I_x cos^2\alpha - I_{xy} sin2\alpha + I_y sin^2\alpha$$

$$I_\eta = \int \xi^2\, dm = \int (y\sin\alpha + x\cos\alpha)^2 dm = I_x sin^2\alpha + I_{xy} sin2\alpha + I_y cos^2\alpha$$

$$D_{\xi\eta} = \int \xi\eta\, dm = \int (y\sin\alpha + x\cos\alpha)(y\cos\alpha - x\sin\alpha) dm$$
$$= D_{xy} cos2\alpha + \frac{1}{2} sin2\alpha\, (I_x - I_y)$$

We know that for the main axes the moment of deviation must be zero. Thus, the equation for angle α_O determining the position of the main inertia axes in relation to the O_{xy} system is obtained.

$$\frac{(I_x - I_y)}{2}\, sin2\alpha_O + D_{xy} cos2\alpha_O = 0$$

$$tan2\alpha_O = \frac{2D_{xy}}{(I_y - I_x)}$$

$$\alpha_O = \frac{1}{2}\arctan\left(\frac{2D_{xy}}{I_y - I_x}\right)$$

After determining the angle α_O, the values of the main moments of inertia can be calculated

$$I_1 = I_{max} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4{D_{xy}}^2}$$

$$I_2 = I_{min} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4{D_{xy}}^2}$$

For the inverse problem, when the directions of the main axes and moments of inertia about these axes are known, and there is a need to determine moments of inertia relative to the system rotated by the angle α we use equations

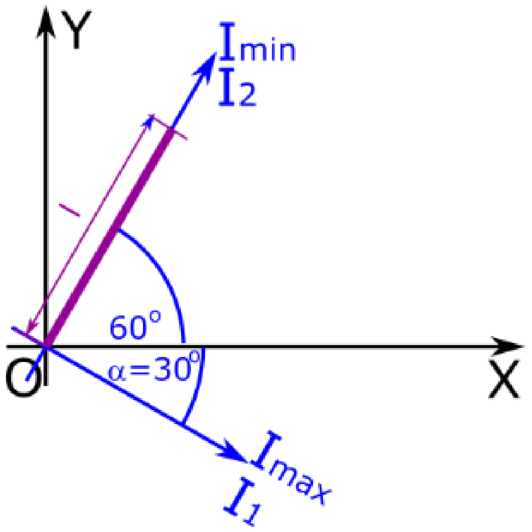
$$I_x = I_1;\ I_y = I_2;\ D_{xy} = 0$$

$$I_\xi = I_1\, cos^2\alpha + I_2\, sin^2\alpha$$

$$I_\eta = I_1\, sin^2\alpha + I_2\, cos^2\alpha$$

$$D_{\xi\eta} = \frac{(I_1 - I_2)}{2}\, sin2\alpha$$

Ex. 1. For the rod with the length l and mass m shown in the figure, located in the main axis system, find the values of the moments of inertia and the moment of deviation in the O_{xy} system.



$$\alpha = 30° = \frac{\pi}{3};\ I_1 = I_{max} = \frac{ml^2}{3};\ I_2 = I_{min} = 0;\ D_{12} = 0;\ I_x, I_y, D_{xy} = ?$$

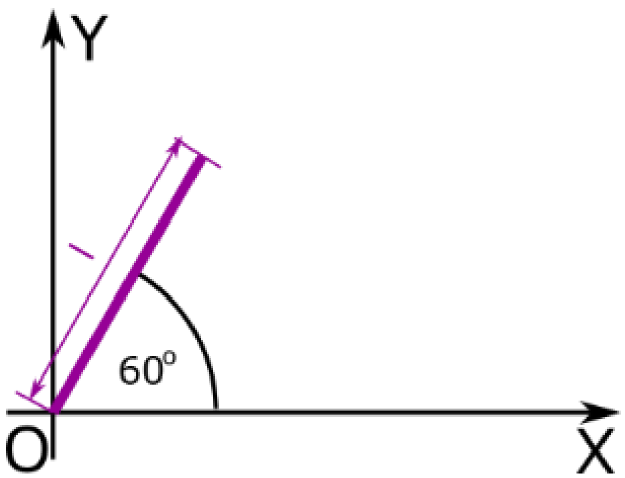
We use the equations shown above

$$I_x = I_1\, cos^2\alpha + I_2\, sin^2\alpha = \frac{ml^2}{3} * \frac{3}{4} = \frac{ml^2}{4}$$

$$I_y = I_1\, sin^2\alpha + I_2\, cos^2\alpha = \frac{ml^2}{3} * \frac{1}{4} = \frac{ml^2}{12}$$

$$D_{xy} = \frac{(I_1 - I_2)}{2}\, sin2\alpha = sin\frac{2\pi}{3} * \frac{\frac{ml^2}{3} - 0}{2} = \frac{\sqrt{3}}{12} ml^2$$

Ex. 2. Inverse problem. The values of the moments of inertia for the O_{xy} axis system are known. Find the values of the main moments and the angle by which the O_{xy} system should be rotated to find the main axis system.



$$\alpha = 30° = \frac{\pi}{3};\ I_x = \frac{ml^2}{4};\ I_y = \frac{ml^2}{12};\ D_{xy} = \frac{\sqrt{3}}{12} ml^2;\ I_{max}, I_{min}, \alpha_O = ?$$

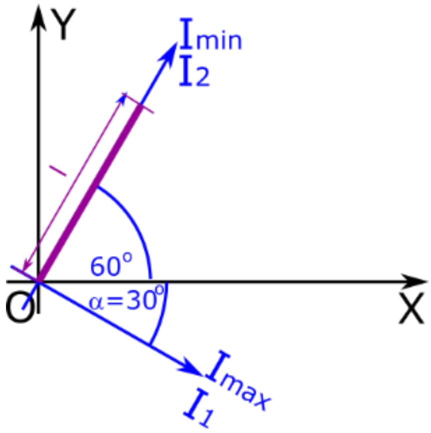
To solve this example, use the first equations of the subject related to rotational transformation.

$$I_1 = I_{max} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2}$$
$$= \frac{1}{2}\left(\frac{ml^2}{4} + \frac{ml^2}{12}\right) + \frac{1}{2}\sqrt{\left(\frac{ml^2}{36}\right)^2 + 4(\frac{\sqrt{3}}{12}ml^2)^2} = \frac{ml^2}{3}$$

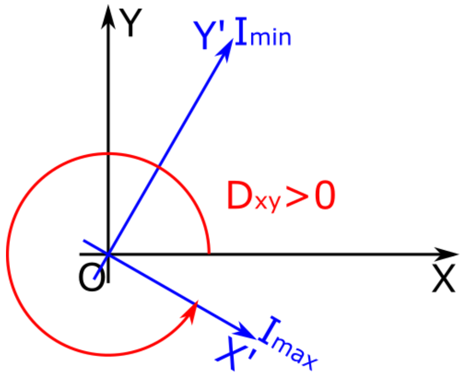
$$I_2 = I_{min} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4D_{xy}{}^2} = \frac{1}{2}\left(\frac{ml^2}{4} + \frac{ml^2}{12}\right)$$
$$- \frac{1}{2}\sqrt{\left(\frac{ml^2}{36}\right)^2 + 4(\frac{\sqrt{3}}{12}ml^2)^2} = 0$$

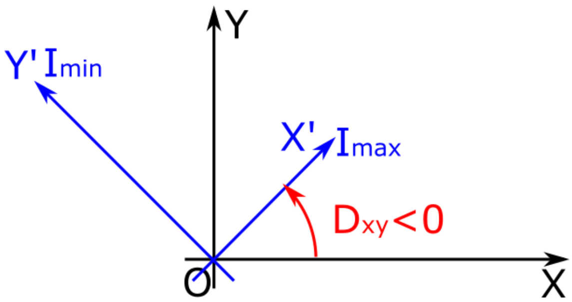
$$\alpha_O = \frac{1}{2}\arctan\left(\frac{2D_{xy}}{I_y - I_x}\right) = \frac{1}{2}\arctan\left(\frac{2\frac{\sqrt{3}}{12}ml^2}{\frac{ml^2}{12} - \frac{ml^2}{4}}\right) = -30°$$

Angle α_O tells us how much we need to rotate our system, taking O_x as the reference axis. Because we have a negative angle, it means that we must rotate our system 30° down.

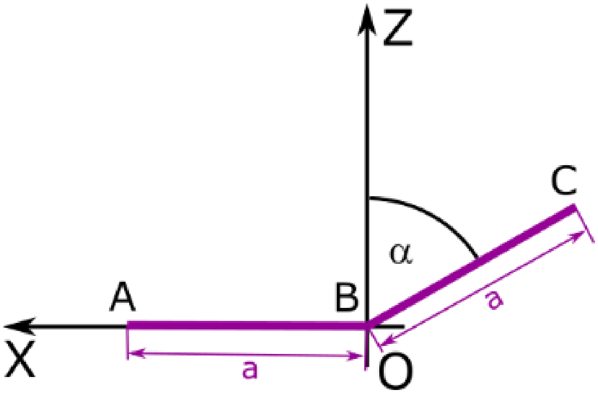


The second important issue is marking which axis is the maximum axis and which is the minimum. To be able to determine this, we need to look at the value of the moment of deviation for our initial system. If the deviation moment is greater than zero, the obtuse angle will be between the axis O_x and the maximum moment of inertia. however, if the deviation moment value is negative, then we have an acute angle between the O_x axis and the maximum axis of the moment of inertia.





Ex. 3. Find the moment of inertia relative the axis "z" for a thin homogeneous ABC rod. Part AB is perpendicular to O_z, and part BC connected to the O_z axis at an angle α. Data: AB = BC = a, α, $m_{AB}=m_{BC}=m$.



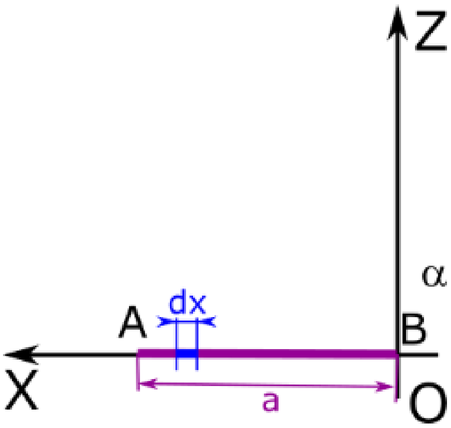
The system can be considered separately as an AB and BC member assembly.

$$I_z = I_{zAB} + I_{zBC}$$

Moment of inertia for segment AB

$$I_{zAB} = \int (x^2 + y^2)\, dm$$

Since we are dealing with a bar, the size along the axis y = 0, therefore



$$I_{zAB} = \int x^2\, dm$$

Now we just need to define what dm is to us.

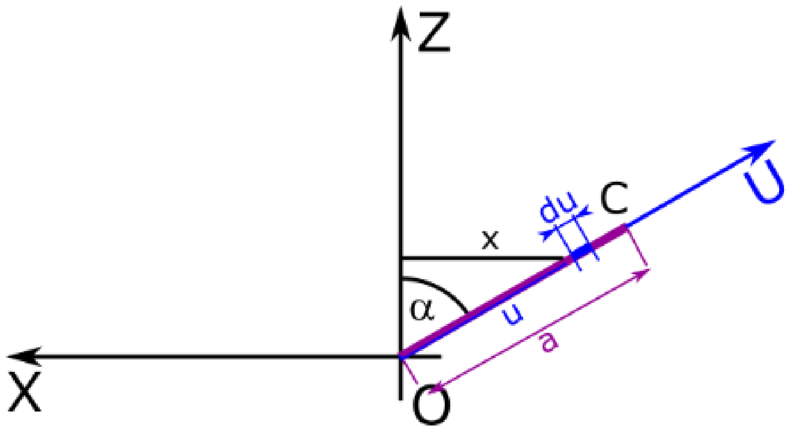
$$dm = \rho dx$$

$$I_{zAB} = \int_0^a x^2\, \rho dx = \frac{1}{3}\rho a^3$$

$$m = \rho a \ \rightarrow \rho = \frac{m}{a}$$

$$I_{zAB} = \frac{1}{3} m a^2$$

Moment of inertia for part BC



$$I_{zBC} = \int (x^2 + y^2)\, dm$$

Since we are dealing with a bar, the size along the axis y = 0, therefore

$$I_{zBC} = \int x^2\, dm$$

In order to make calculations easier, let's introduce the O_u axis against which we will perform calculations.

Now we just need to define what dm is to us.

$$dm = \rho du$$

Because our output integral is related to values on the x axis, we must express these values on the u axis.

$$\frac{x}{u} = sin\alpha \ \rightarrow x = usin\alpha$$

Hence

$$I_{zBC} = \int x^2\, dm = \int (usin\alpha)^2\, dm = \int_0^a u^2\, sin^2\alpha\rho du = \frac{1}{3}\rho a^3 sin^2\alpha$$

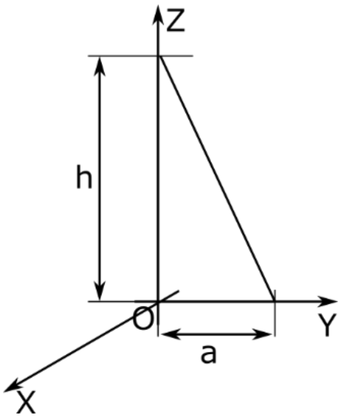
$$m = \rho a \ \rightarrow \rho = \frac{m}{a}$$

$$I_{zBC} = \frac{1}{3} ma^2 sin^2\alpha$$

$$I_z = I_{zAB} + I_{zBC}$$

$$I_z = \frac{1}{3} ma^2 + \frac{1}{3} ma^2 sin^2\alpha = \frac{1}{3} ma^2 (1 + sin^2\alpha)$$

Ex. 4. Calculate the axial moments of inertia I_x, I_y, I_z and the moments of deviation D_{xy}, D_{yz}, D_{xz} from a homogeneous triangular plate with a mass m of base equal to a and height h.

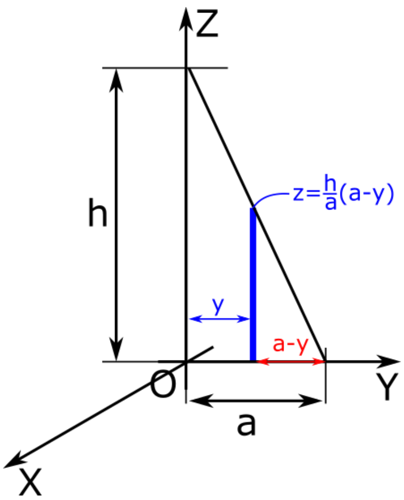


The x axis is perpendicular to the symmetry plane of the board, so it is one of the main axes of the system. Hence $D_{xz} = D_{xy} = 0$.

One more moment of deviation remains to be calculated. We will calculate this moment by definition.

$$D_{yz} = \int yzdm = \rho \iint yzdydz$$

At this point, one should wonder how the coordinate values change after the y axis and after the z axis. If we assume that the values on the y axis change from 0 to a, then using the y-axis function we should describe the change of coordinates on the z axis.



$$D_{yz} = \int yzdm = \rho \iint yzdydz = \rho\left(\int_0^a ydy \int_0^{\frac{h}{a}(a-y)} zdz\right) = \rho\frac{a^2h^2}{24}$$

$$\rho = \frac{2m}{ah}$$

$$D_{yz} = \frac{mah}{12}$$

Moments of inertia about the x, y, z axis.

$$I_x = \int (z^2 + y^2)\, dm = \rho \int_0^a y^2 dy \int_0^{\frac{h}{a}(a-y)} dz + \rho \int_0^a dy \int_0^{\frac{h}{a}(a-y)} z^2 dz = \frac{mah}{3}$$

$$I_y = \int (x^2 + z^2)\, dm = \int z^2\, dm = \rho \int_0^a dy \int_0^{\frac{h}{a}(a-y)} z^2 dz = \frac{mh^2}{6}$$

$$I_z = \int (x^2 + y^2)\, dm = \int y^2\, dm = \rho \int_0^a y^2 dy \int_0^{\frac{h}{a}(a-y)} dz = \frac{ma^2}{6}$$