Kinematics of the particle (material point)

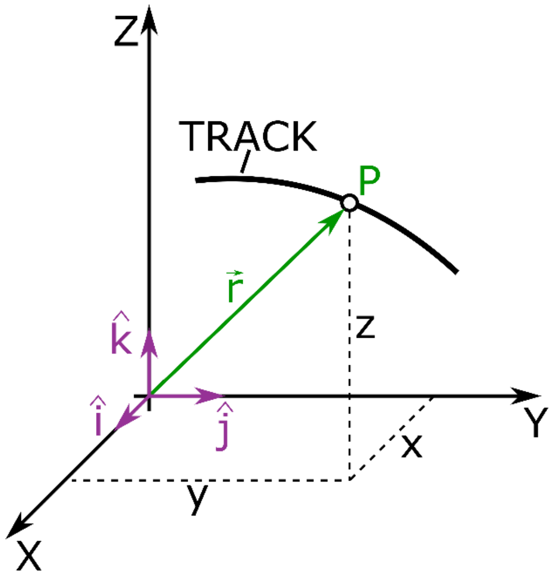
Kinematics is a part of mechanics dealing with the movement of bodies, without entering into the relationship between the movement of the examined body (in particular of the point) and the forces acting on it.

In the case of kinematics, we will consider what happens to the body in space over time. We will describe this type of relationship as the geometry of motion.

Movement of the body - changing the position of this body in relation to another one taken from a stationary body (reference body). In the case of mechanics, the Earth is usually taken as the reference body.

Reference system - a system that is fixed and bound to a reference body. The most common reference system is a rectangular coordinate system (Euclidean space).

Euclidean space



x, y, z-coordinates of the moving point P with respect to the fixed coordinate system (reference system).

In order to describe the movement of this point, it is necessary to determine how particular coordinates change with time.

$$x = f_1(t);\ y = f_2(t);\ z = f_3(t)$$

We will call the above equations the kinematic equations of motion.

Point TRACK - line along which point P moves in space.

Parametric equation of the point track - in the equation the time is a parameter. After removing time, we get the relations between the x, y, z coordinates (i.e. the path of motion).

We can also describe the motion of a point in terms of a vector radius $\vec{r}$ from time $\vec{r} = \vec{r}(t)$.

Vector components

$$\vec{r} = \vec{r}_x + \vec{r}_y + \vec{r}_z = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$$

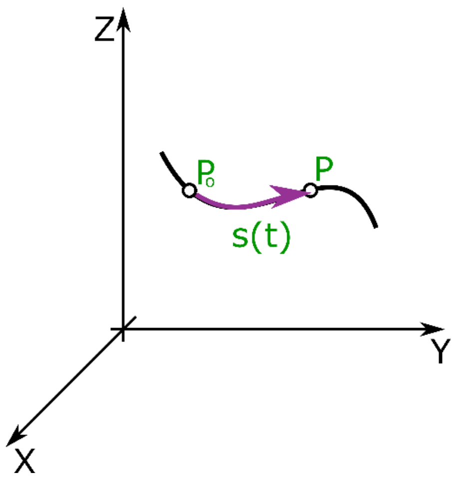
$$r_x = x(t); \quad r_y = y(t); \quad r_z = z(t)$$

$$\vec{r} = \hat{\imath}(t) + \hat{\jmath}(t) + \hat{k}(t)$$

Where $\hat{\imath}$; $\hat{\jmath}$; $\hat{k}$ are the versors of the reference coordinate system.

We know very well that we do not always have to move only in a rectangular system, and in some cases it is better to operate in a different coordinate system. So let's start by describing the movement of a point on the path using an arc coordinate.

When the path of a moving point P is known, it is possible to describe the position of this point by specifying the coordinate s measured along the path from a full stationary point P_o.



s - arc coordinate equal to the arc length P_oP,

When the point P moves, then s is a function of time

$$s = f(t)$$

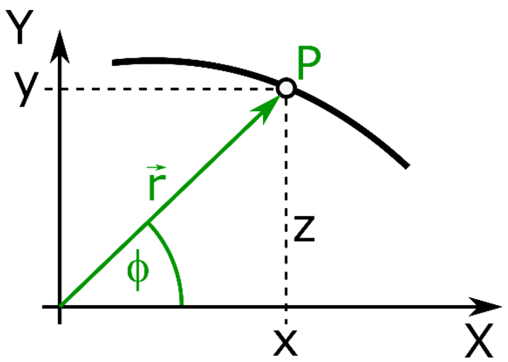
Equation of motion of a point on a track

Equations of motion of a point in curvilinear coordinates

In addition to rectangular coordinates, the path of a point can be defined by curvilinear coordinates.

Polar system on a plane

Let's start with a polar system on a plane. In such a system, a point moves only in one plane, and its instantaneous position can be determined by specifying the length of the leading radius $\vec{r}$ and the angle ϕ with the polar axis. The polar axis is the axis for which ϕ = 0.

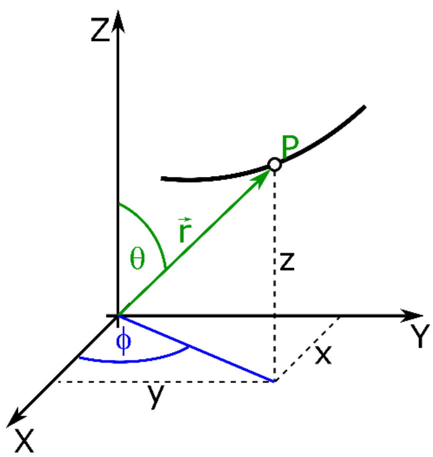


$$r = f_1(t);\ \phi = f_2(t)$$

Transition from polar coordinates to the Cartesian system	Transition from the Cartesian system to the polar system
$x = r\cos\phi$ $y = r sin\phi$	$r = \sqrt{x^2 + y^2}$ $\phi = arctan\frac{y}{x}$

Polar system in space (spherical)

The position of the point is described by the vector $\vec{r}$, its length and angles θ and ϕ.
Equation of motion



$$r = f_1(t);\ \phi = f_2(t);\ \theta = f_3(t)$$

Transition from spherical coordinates to the Cartesian system

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\ sin\phi$$
$$z = r\cos\theta$$

Transition from the Cartesian system to the spherical system

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = arccos\frac{y}{x}$$
$$\theta = arccos\frac{z}{r}$$

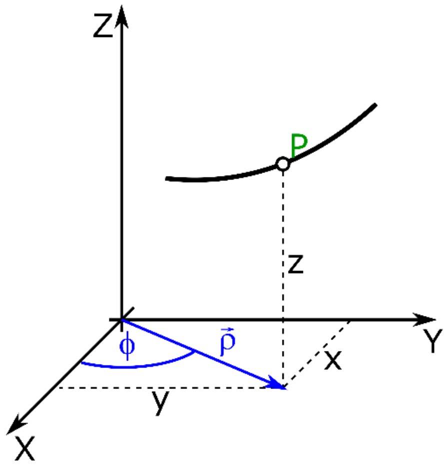
<u>Cylindrical coordinate system</u>

Position of the point defined by:

z - position coordinate

ρ - distance from the z axis

ϕ - angle



$$z = f_1(t);\ \rho = f_2(t);\ \phi = f_3(t)$$

Transition from cylindrical coordinates to the Cartesian system

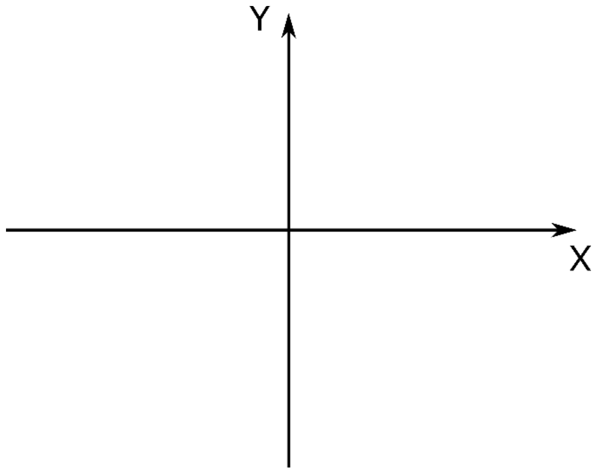
$$x = \rho\cos\phi$$
$$y = \rho sin\phi$$
$$z = z$$

Transition from the Cartesian system to the cylindrical system

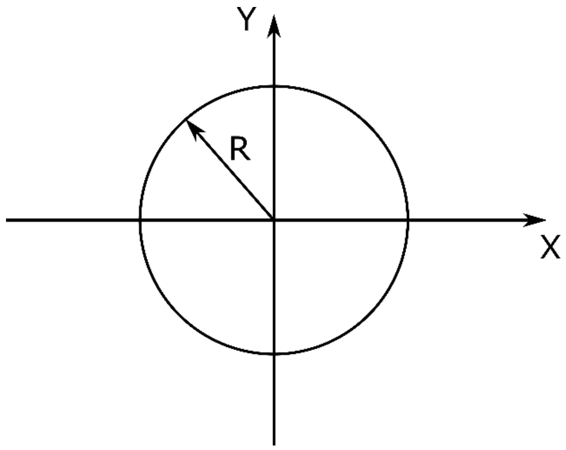
$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = arctan\frac{y}{x}$$
$$z = z$$

Example 1. The path of the point P is a circle with the radius R. Describe the movement of this point using the radius vector $\vec{r}$. Take the center of the circle at the center of the coordinate system.

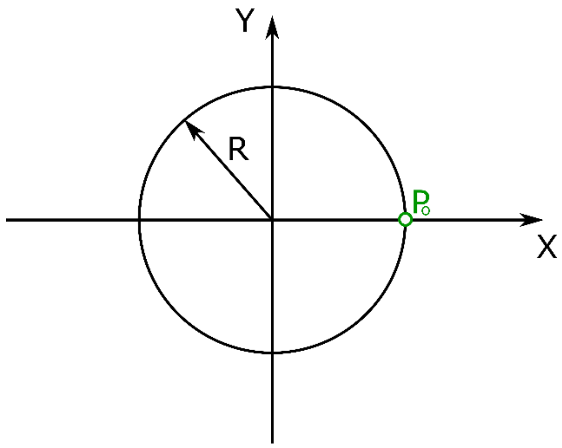
We will begin the solution of the task by drawing a coordinate system.

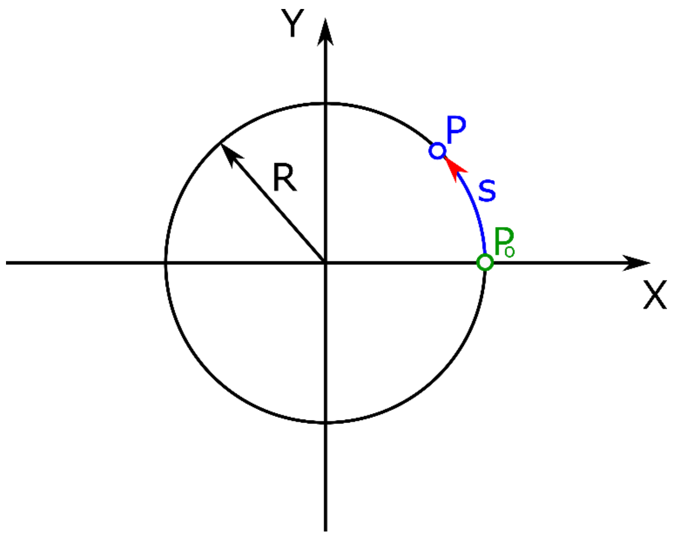


Then, according to the data in the problem, let's place the center of the circle in the center of our coordinate system.



Assume that the point started to move from the point P_o, which lies on the axis X.





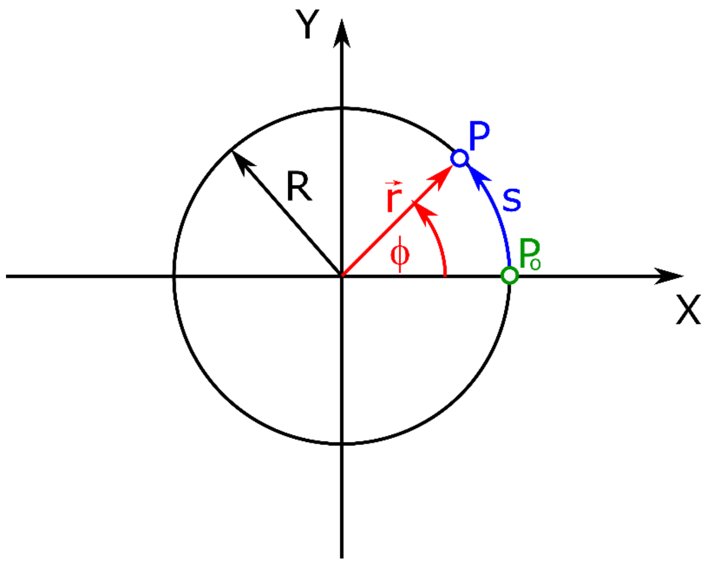
The instantaneous position of the point is determined by specifying the arc coordinate s equal to the arc length P_oP.

We know that

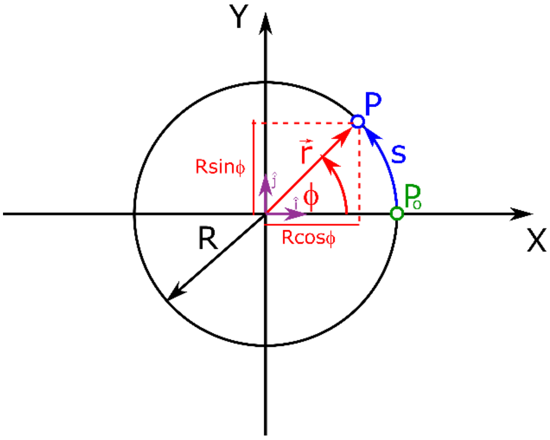
$$s = r * \phi$$

$$s = r * \phi(t)$$

$\phi(t)$ – the angle of rotation of the radius vector



now going from Cartesian coordinates we get



$$r = \sqrt{x^2 + y^2}$$

$$\phi = arctan\frac{y}{x}$$

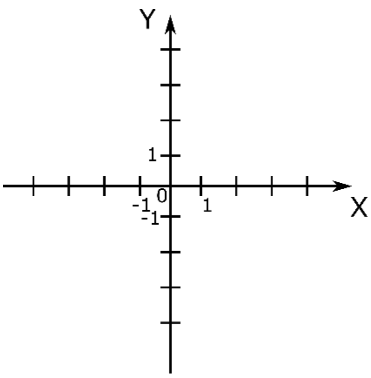
Finally, we can write

$$s = r * \phi$$

$$s = \sqrt{x^2 + y^2} * \ arctan\frac{y}{x}$$

Example 2. Given are the equations of motion of the point P moving in the Oxy plane. $x = 3 + 2t; y = -2t$, determine the trajectory of the point.

At the beginning, as in the previous task, let's insert a coordinate system.



Next, let's find the equation of the trajectory. We can see that the equations given in the problem are parametric equations of motion, thanks to which we know what the position of the point is for a given time. Therefore, our variable parameter is time. To find an equation that shows us the trajectories of a point's motion, we need to get rid of time from the given equations.

$$t = -\frac{y}{2} \rightarrow x = 3 + 2\left(-\frac{y}{2}\right) = 3 - y$$

We can write the point path equation as follows

$$x = 3 - y$$

or

$$y = 3 - x$$

Let's check where the point is at time t = 0

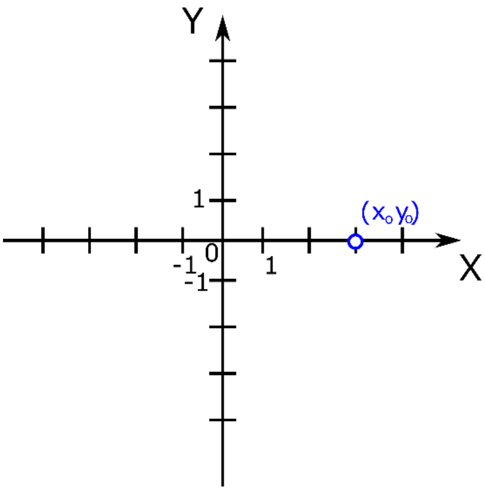
$$x = 3 + 2t$$

$$y = -2t$$

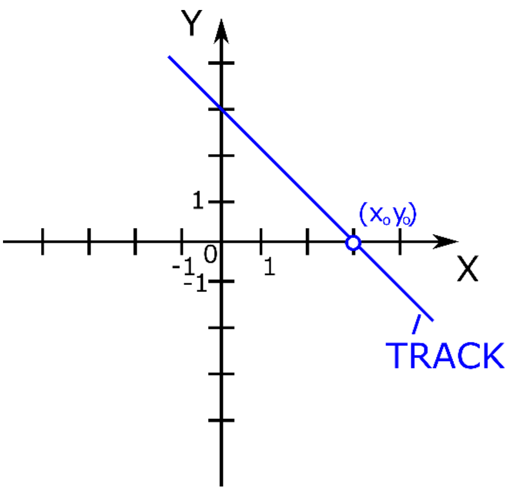
For this purpose, in the given parametric equations for motion, we change the time t to a value equal to 0.

$$x(t = 0) = 3$$

$$y(t = 0) = 0$$



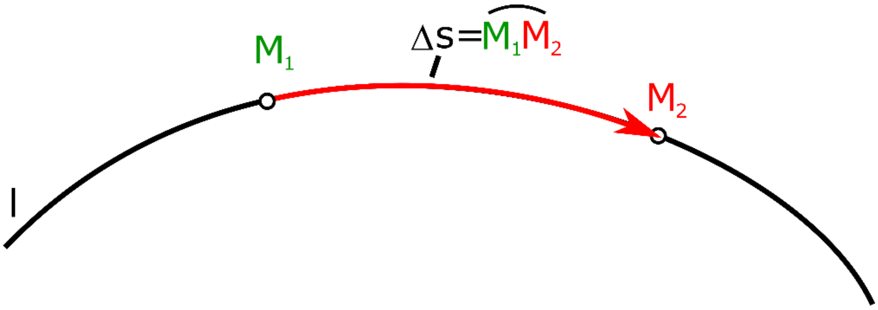
Finally, let's also put the designated path of our point on the graph.



VELOCITY AND ACCELERATION

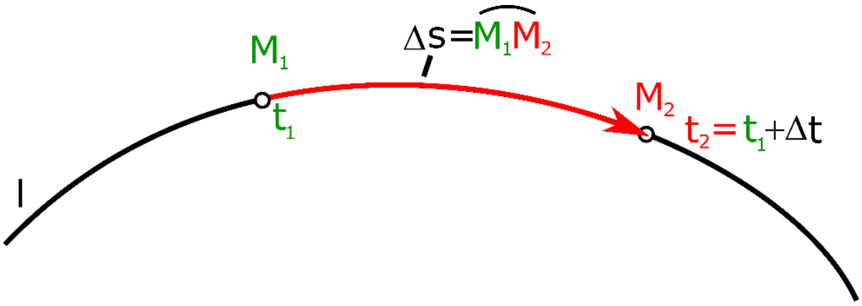
Velocity

We begin our deliberations with the velocity of the material point. Let us consider the movement of point M from point M_1 to point M_2. It can be seen that the path covered by the point equals some Δs equal to the arc length M_1M_2.

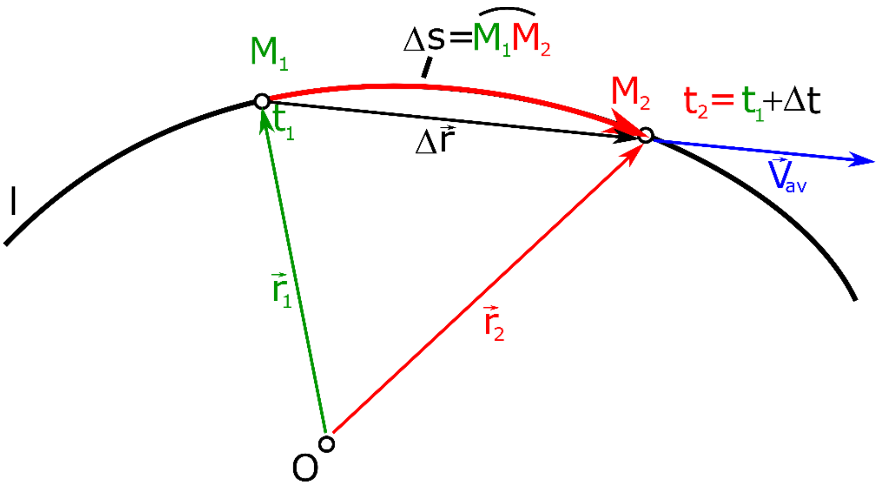


We will assume that the point is at M_1 at time t_1 and at M_2 at time t_2, where:

$$t_2 = t_1 + \Delta t$$



The position of the point in M_1 and M_2 can be described with the help of the vector $\vec{r}_1$ and $\vec{r}_2$. It can be seen that the determination of the geometric increase of the vector r will be of significant importance for determining the change in position.

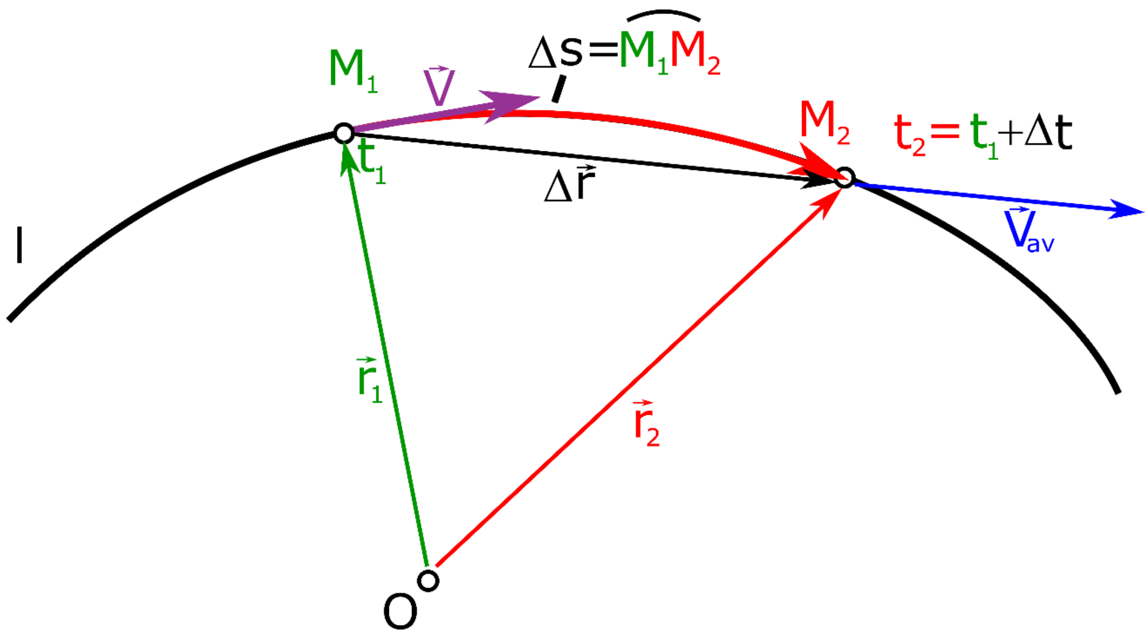


This increase can be written as follows.

$$\Delta\vec{r} = \vec{r}_2(t_2) - \vec{r}_1(t_!)$$

The ratio of the vector $\vec{r}$ increase to the time in which this increase took place is called the average speed.

$$\vec{V}_{av} = \frac{\Delta\vec{r}}{\Delta t}$$



The average velocity vector $\vec{V}_{av}$ has a chord direction. In all practical measurements we always determine the average value, which depends on the distance between points M_1 and M_2. It depends on the point's movement and the choice of points on the movement path.

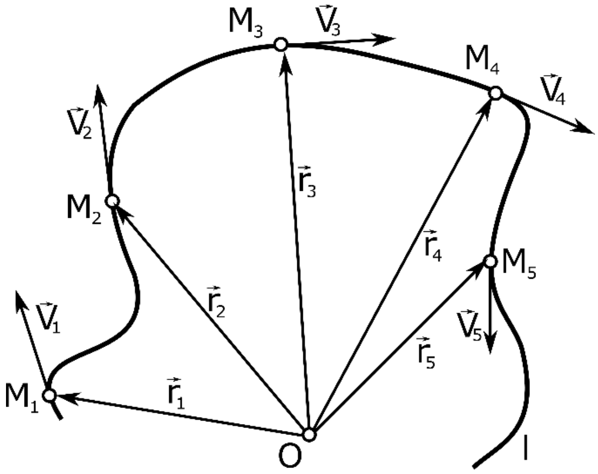
Besides the average speed, there is the concept of instantaneous velocity $\vec{V}$. The instantaneous velocity vector will exist if the radius $\vec{r}$ is differentiable. It is an abstract concept, but it is of great importance and uniquely characterizes the movement at a given moment.
If we assume that $\Delta t \to 0$ and $\Delta s \to min$, then the chord will go to the tangent. Hence the velocity vector will also be tangent to the path of motion.

Instantaneous velocity

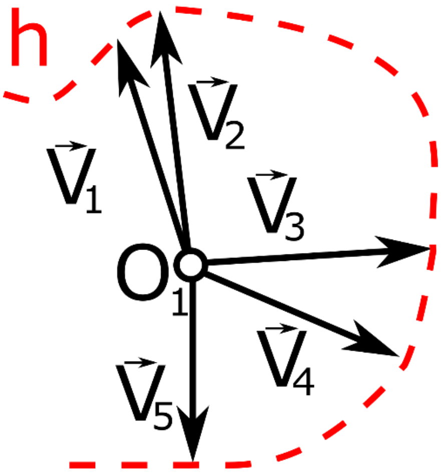
$$\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t)$$

HODOGRAPH OF VELOCITY

Let us assume that the path l of a moving point M describes the end of the vector $\vec{r}$ which beginning is a point O. The velocities $\vec{V_i}$ at successive points M_i are tangent to this path of motion.

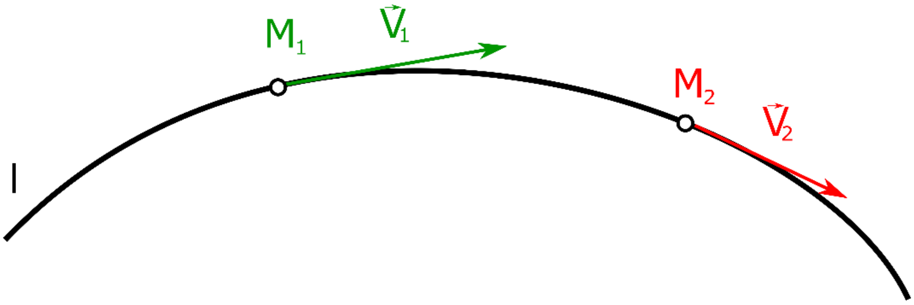


If we move the velocity vectors parallel to the common point O_1, then the ends of these vectors will lie on the line marked h, called the HODOGRAPH of the velocity of a given point M.



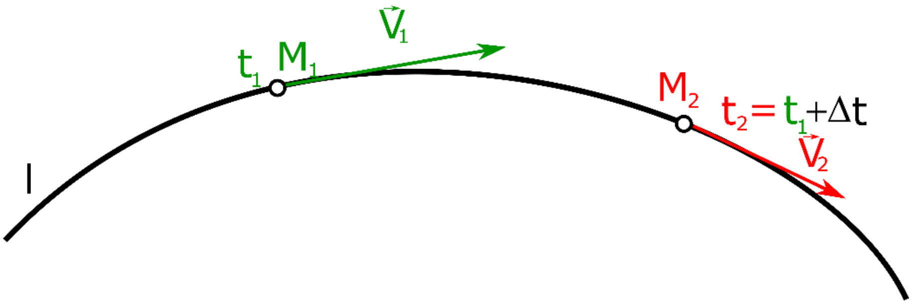
Acceleration

Let us assume that the point follows the curve l, with the velocity $\vec{V}_1$ at M_1 and the velocity $\vec{V}_2$ at M_2.

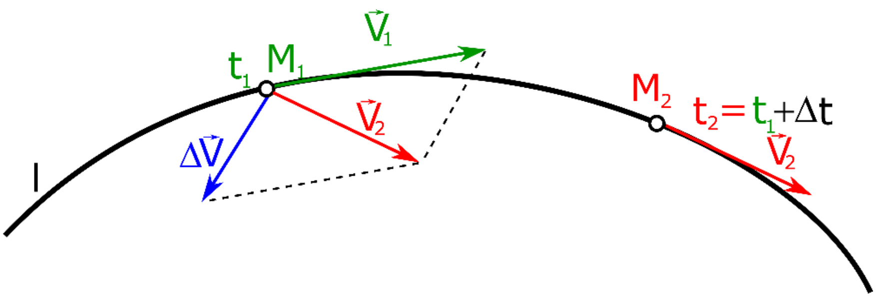


We will assume that the point is at M_1 at time t_1 and at M_2 at time t_2, where:

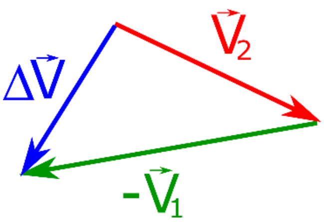
$$t_2 = t_1 + \Delta t$$



Velocity increases between points M_1 and M_2.



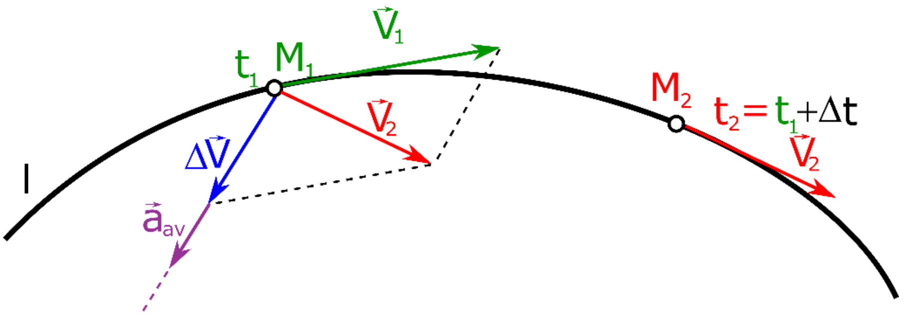
$$\Delta\vec{V} = \vec{V}_2 - \vec{V}_1$$



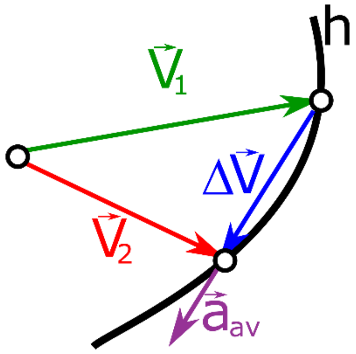
The ratio of the vector $\vec{V}$ increase to the time in which this increase took place is called the average acceleration.

$$\vec{a}_{av} = \frac{\Delta\vec{V}}{\Delta t}$$

$\vec{a}_{av}$ has the direction of velocity increase $\Delta\vec{V}$, whereby its value and return depend on the time interval of its determination Δt.



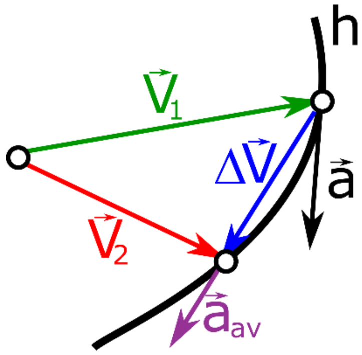
Besides the average acceleration, there is also an instantaneous acceleration $\vec{a}$.
In order to determine the instantaneous acceleration, we use the velocity hodograph



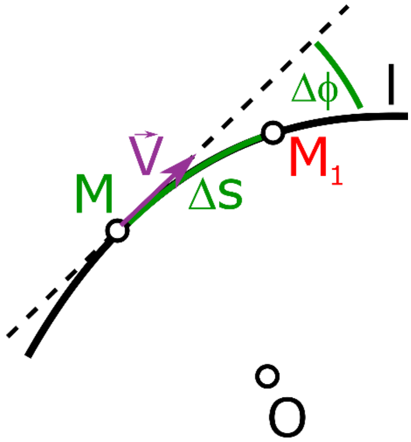
The instantaneous acceleration vector is directed along the tangent to the velocity hodograph.

Instantaneous acceleration

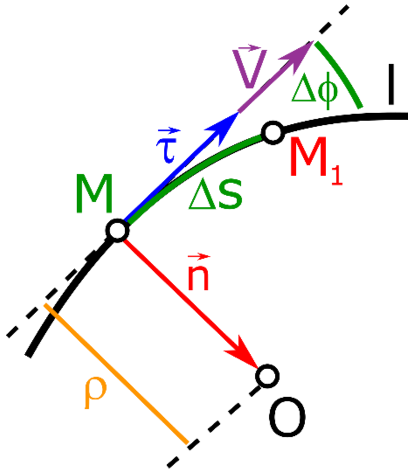
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{V}}{\Delta t} = \frac{d\vec{V}}{dt} = \dot{\vec{V}}(t) = \ddot{\vec{r}}(t)$$



CURVILINEAR MOVEMENT



When a point's path is a plane curve, its natural directions are tangent and normal.



ρ - radius of curvature lying on the line of the normal unit vector $\hat{n}$.

$\vec{V}$ - velocity on the line of the tangential unit vector $\hat{\tau}$

$$\rho = \frac{1}{C}$$

C – curvature,

$$C_{av} = \frac{\Delta\phi}{\Delta s}$$

C_{av} – average curvature of MM_1 curve

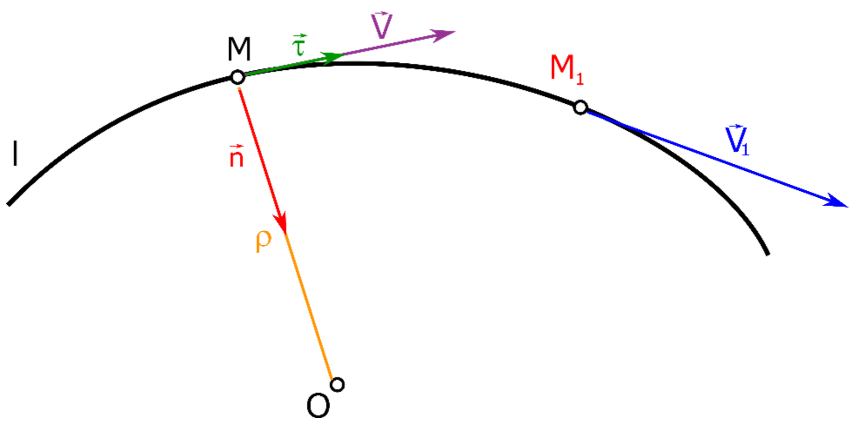
curvature at a point

$$C = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta s} = \frac{\mathrm{d}\phi}{\mathrm{d}s}$$

TANGENTIAL AND NORMAL ACCELERATION

The acceleration $\vec{a}$ of a point M moving along a spatial curve must lie in a strictly tangent plane, because as a derivative of the velocity $\vec{V}$, it is tangent to the velocity hodograph of this point. Moreover, the velocity vector is always tangent to the curve along which the point moves.

It is assumed that the point follows the curve from M to M_1. It has a velocity $\vec{V}$ at M and a velocity $\vec{V}_1$at M_1. Let us introduce two unit vectors into the system, a tangent $\hat{\tau}$, lying in the velocity direction $\vec{V}$, and a normal one $\hat{n}$, directed to the center of the curvature.

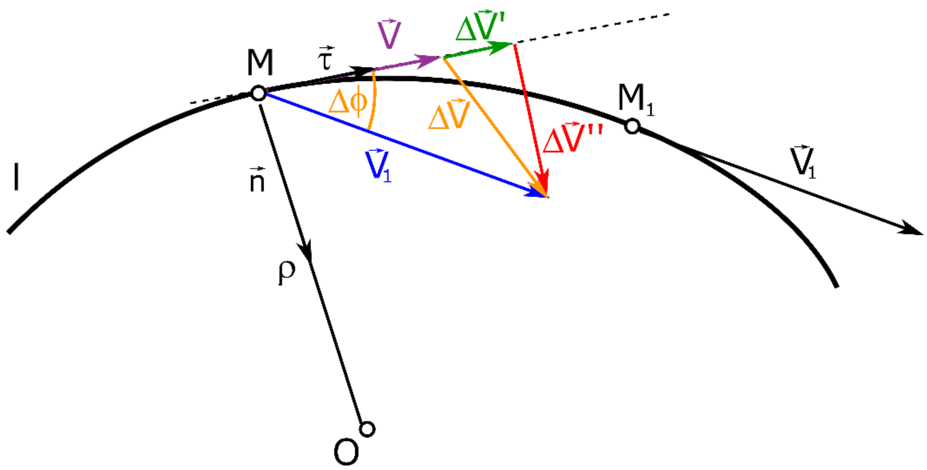


The velocity vector gain will be

$$\Delta\vec{V} = \vec{V}_1 - \vec{V}$$

transforming the expression we will get,

$$\vec{V}_1 = \vec{V} + \Delta\vec{V}$$



Further, we can see that the$\Delta\vec{V}$ vector can also be written as the sum of two vectors $\Delta\overrightarrow{V'}$and $\Delta\overrightarrow{V''}$, which will lie on the tangent and normal directions respectively.

$$\Delta\vec{V} = \Delta\overrightarrow{V'} + \Delta\overrightarrow{V''} = \Delta V'\hat{\tau} + \Delta V''\hat{n}$$

Earlier we wrote that acceleration $\vec{a}$ is equal to:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t}$$

We will now write this equation using the introduced vectors.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{V'}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{V"}}{\Delta t} = \hat{\tau} \lim_{\Delta t \to 0} \frac{\Delta V'}{\Delta t} + \hat{n} \lim_{\Delta t \to 0} \frac{\Delta V"}{\Delta t} = \hat{\tau} a_\tau + \hat{n} a_n$$

The above equation can generally be written as follows.

$$\vec{a} = \vec{a}_\tau + \vec{a}_n$$

The total acceleration is then the sum of the tangential and normal acceleration.

Based on the above conclusion, let's try to write both components of acceleration with the velocities as given in points M and M_1.

$$\Delta V' = V_1 \cos \Delta \phi - V$$

$$\Delta V" = V_1 \sin \Delta \phi$$

Let's start with the component in the tangential direction

$$\vec{a}_\tau = \hat{\tau} \lim_{\Delta t \to 0} \frac{\Delta V'}{\Delta t} = \hat{\tau} \lim_{\Delta t \to 0} \frac{V_1 \cos \Delta \phi - V}{\Delta t} = \hat{\tau} \lim_{\Delta t \to 0} \frac{V_1 - V}{\Delta t} = \frac{dV}{dt} \hat{\tau}$$

$$\text{if } \Delta \phi \to 0 \; then \; \cos \Delta \phi \approx 1; \qquad V_1 - V = \Delta V$$

The final equation will be

$$\vec{a}_\tau = \frac{dV}{dt} \hat{\tau}$$

Once we know how we can find the tangential acceleration, let's do the same for the normal component of the acceleration.

$$\text{if } \Delta \phi \to 0 \; then \; \sin \Delta \phi \approx \Delta \phi;$$

$$\vec{a}_n = \hat{n} \lim_{\Delta t \to 0} \frac{\Delta V"}{\Delta t} = \hat{n} \lim_{\Delta t \to 0} \frac{V_1 \sin \Delta \phi}{\Delta t} = \hat{n} \lim_{\Delta t \to 0} \frac{V_1 \Delta \phi}{\Delta t} = \hat{n} \lim_{\Delta t \to 0} V_1 \frac{\Delta \phi}{\Delta t} * \frac{\Delta s}{\Delta s}$$
$$= \hat{n} \lim_{\Delta t \to 0} V_1 * \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} * \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

We can see that:

$$\lim_{\Delta t \to 0} V_1 = V;$$

$$\lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \frac{1}{\rho};$$

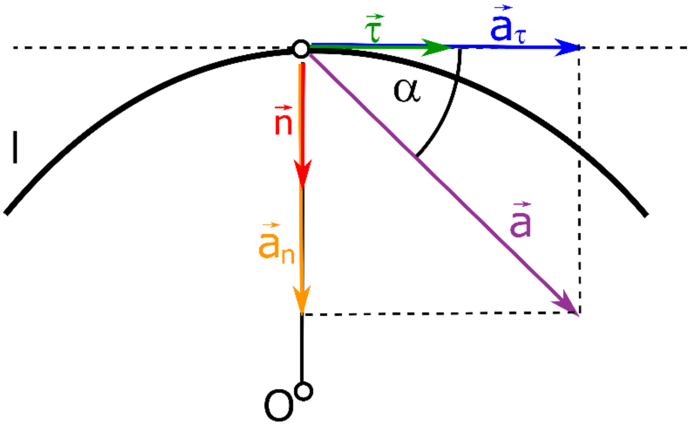
$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = V;$$

Then:

$$\vec{a}_n = \hat{n} \lim_{\Delta t \to 0} V_1 * \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} * \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \hat{n} * V * \frac{1}{\rho} * V = \hat{n}\frac{V^2}{\rho}$$

Ultimately, taking both components of total acceleration into account, we get the following formula

$$\vec{a} = \vec{a}_\tau + \vec{a}_n = \frac{dV}{dt}\hat{\tau} + \hat{n}\frac{V^2}{\rho}$$



$$a = \sqrt{a_\tau^2 + a_n^2} = \sqrt{\left(\frac{dV}{dt}\right)^2 + \frac{V^4}{\rho^2}}$$

$$\sin\alpha = \frac{a_n}{a}$$

$$\cos\alpha = \frac{a_\tau}{a}$$

ACCELERATION AND VELOCITY IN A RECTANGULAR COORDINATE SYSTEM

Parametric equations of motion will take the following form.

$$x = f_1(t);\ y = f_2(t);\ z = f_3(t)$$

In order to obtain the velocity, one must differentiate once the above parametric equations of motion. Then we get projections of the velocity vector on the appropriate axes of the coordinate system.

$$\dot{x} = V_x = \frac{dx}{dt}$$

$$\dot{y} = V_y = \frac{dy}{dt}$$

$$\dot{z} = V_z = \frac{dz}{dt}$$

$$\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$

Then it is enough to calculate the velocity vector modulus according to the equation.

$$V = |\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Further, in order to determine the acceleration, one should differentiate the previously obtained equations of the projections of velocity on individual axes. In this way, we will obtain projections of the acceleration vector on the appropriate axes of the coordinate system.

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$$

$$a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2} = \ddot{y}$$

$$a_z = \frac{dV_z}{dt} = \frac{d^2z}{dt^2} = \ddot{z}$$

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

Then it is enough to calculate the acceleration vector modulus according to the equation.

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

To fully determine the acceleration of a point, one must also find the tangent and normal values of the total acceleration. Below are the calculation of acceleration components for the plane system.

$$a_\tau = \frac{|dV|}{dt} = \dot{V} = \frac{2\dot{V}_x V_x + 2\dot{V}_y V_y}{2\sqrt{V_x^2 + V_y^2}} = \frac{a_x V_x + a_y V_y}{V}$$

$$a_n = -\frac{1}{V}(a_y V_x - a_x V_y)$$