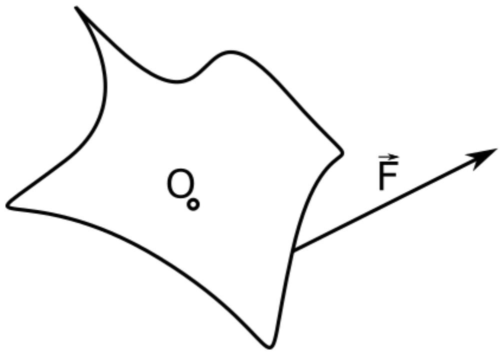
Moment of the force. Divergent system of forces.

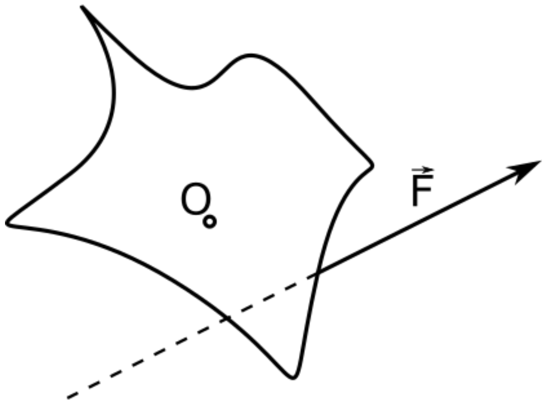
Let's start to talk about moment of force which will be very important in further part of course of Mechanics I and later on Mechanics II.

Moment of a force around a point.

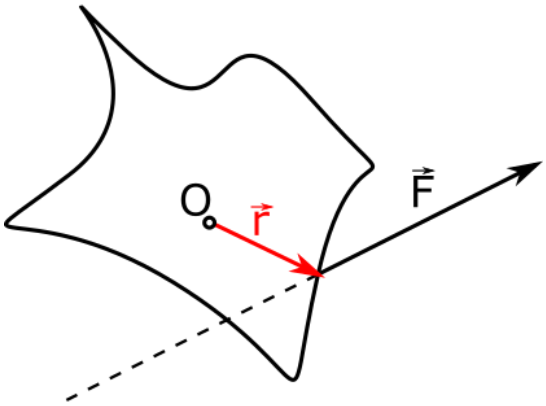
1. Let's assume that we have a body, on which a force F is acting, in a way that is given in the figure.



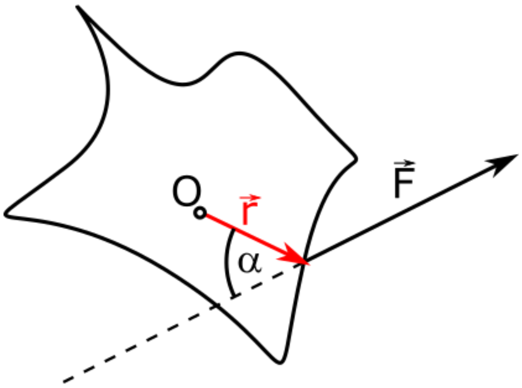
2. One can see that the direction of this force is not going through the O point which is interesting for us. It can be said that force F is acting on point O not directly but indirectly.



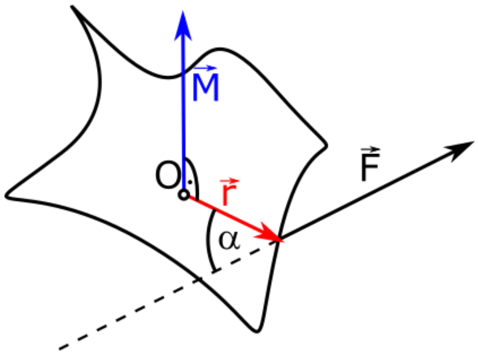
3. Due to the fact that this force is acting indirectly we need to know how we can find influence of the force on point O. Let introduce vector r, which is connecting point O and the force which is acting on it. We can call vector r as an "arm".



4. It can be clearly seen that there is some angle α between vector r and vector of force F.



5. The vector product of vectors r and F will be called as a moment M of a vector F related to point O. The resultant vector of mentioned vector product will be perpendicular to the plane on which vector r and F are placed, and the sense of this vector will depends if we are rotating clockwise or anticlockwise.



$$\vec{M}_o(\vec{F}) = \vec{r} \times \vec{F}$$

The moment of the vector relative to the point will be equal zero if:

- $|\vec{F}| = 0$,
- $\vec{r} \,||\, \vec{F}$,
- direction of acting force F is going through the point related to which we want to calculate moment. In this case point O.

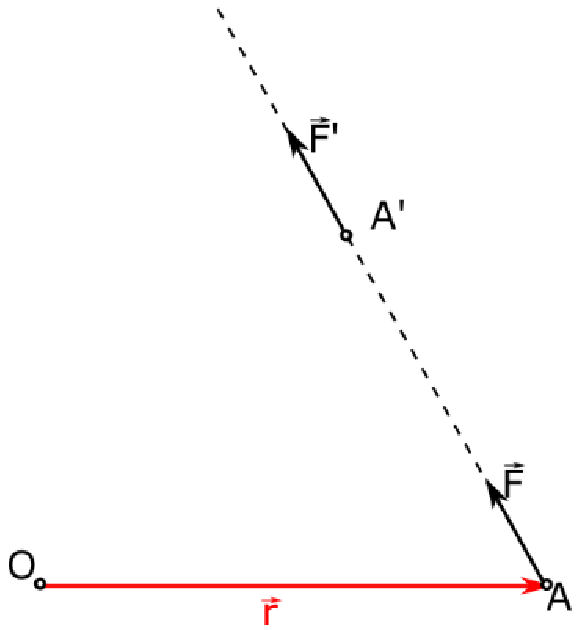
6. Let's see how the moment will change if we move vector along its direction. We have force which is acting somewhere on the point O, but it is acting indirectly.

7. First of all let's introduce vector r which will be attached to our point O and to a force F at point A.

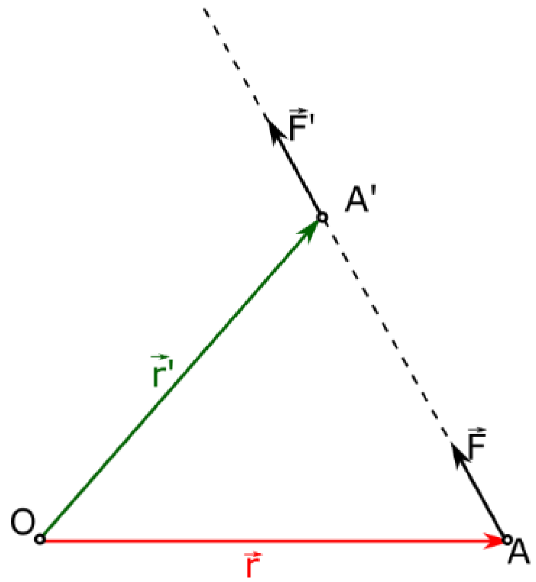
Based on this we can write simple equation of moment

$$\vec{M}_o(\vec{F}) = \vec{r} \times \vec{F}$$

8. But how this equation will change if we will move our vector F from point A along its direction to point A'. Let's call this new vector as F'.



9. If we want to calculate moment for this new position of vector of force we need to introduce new vector r' in order to have "arm" vector.



Now we can write new equation of moment

$$\vec{M}_o(\vec{F}') = \vec{r'} \times \vec{F'}$$

It is very similar to previous one. We can notice that our vector r' we can write as a sum of two other vectors r and AA'.

$$\vec{r'} = \vec{r} + \overrightarrow{AA'}$$

Let's introduce this relationship, to the equation of moment

$$\vec{M}_o(\vec{F}') = \overrightarrow{(r + AA')} \times \vec{F'} = \vec{r} \times \vec{F'} + \overrightarrow{AA'} \times \vec{F'}$$

$$\vec{F} = \vec{F'}$$

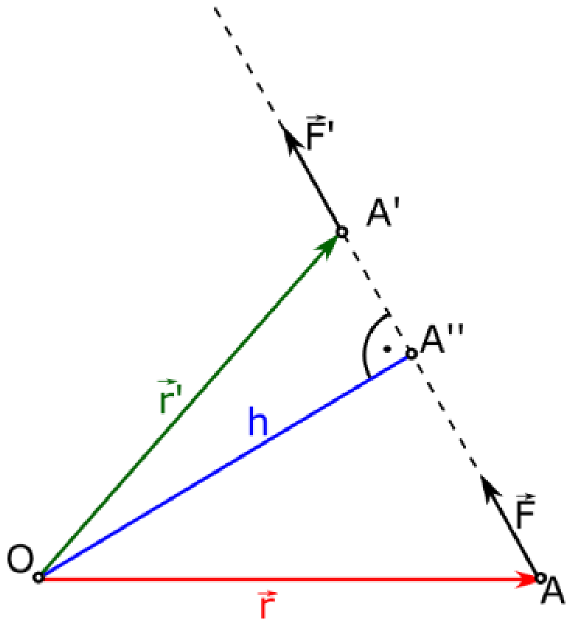
$$\overrightarrow{AA'} \times \vec{F'} = 0 \; because \; \overrightarrow{AA'} || \vec{F'}$$

$$finally \; we \; get$$

$$\vec{M}_o(\vec{F}') = \vec{r} \times \vec{F} = \vec{M}_o(\vec{F})$$

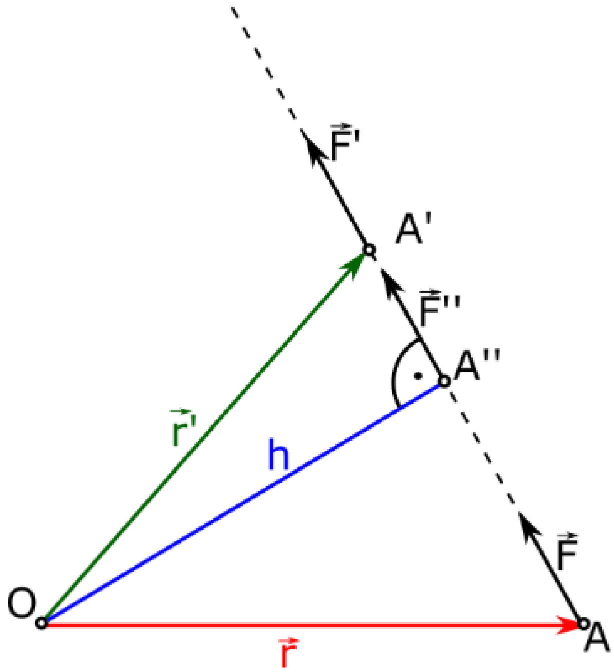
Based on above information we can see that moment of the vector relative to the point would not change if we move this vector along its direction.

10. Let's move vector F to point A'' so that the vector of "arm" will be perpendicular to the direction over which our vector is placed.



In this situation we call vector of the "arm" as h.

11. Now we can move vector F to point A'' and we know that it would not change the value of the moment.



Equation of the moment will have such formula:

$$\vec{M}_O(\vec{F}) = \overrightarrow{OA} \times \vec{F}$$
$$|\vec{M}_O(\vec{F})| = |\vec{F}| * h$$

12. The moment of the vector F relative to the point can be expressed using the coordinates of the vector F given in a rectangular coordinate system.

$$\vec{r}_A = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

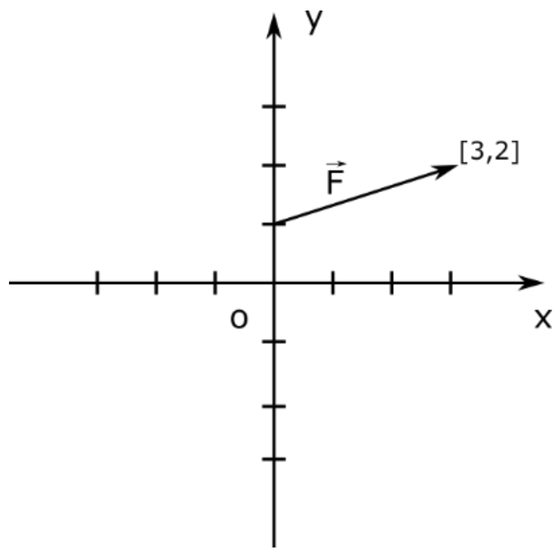
$$\vec{M}_o(\vec{F}) = \vec{r}_A \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (yF_z - zF_y)\hat{\imath} + (zF_x - xF_z)\hat{\jmath} + (xF_y - yF_x)\hat{k}$$

$$\vec{M}_o(\vec{F}) = M_{Ox}\hat{\imath} + M_{Oy}\hat{\jmath} + M_{Oz}\hat{k}$$

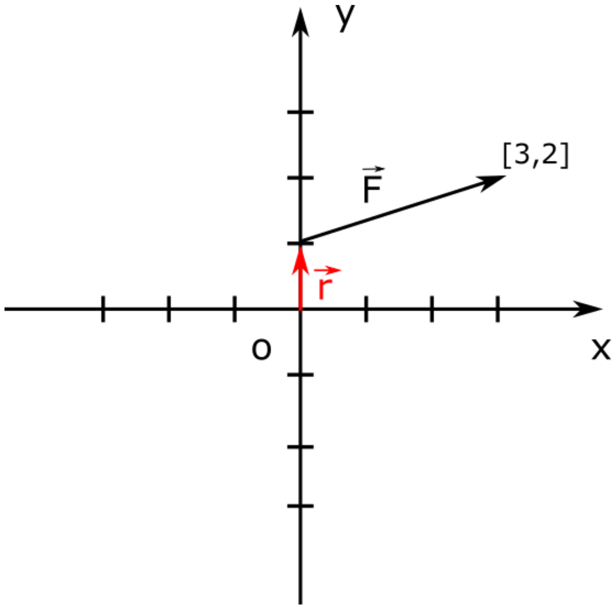
$$\left.\begin{aligned} M_{Ox} &= yF_z - zF_y \\ M_{Oy} &= zF_x - xF_z \\ M_{Oz} &= xF_y - yF_x \end{aligned}\right\}$$

Example 1.

Find moment from the force F relative to the beginning of coordinates system, if he force is in given position.



1. First we need to introduce vector from the beginning of coordinates system to the beginning of force, as we did previously.

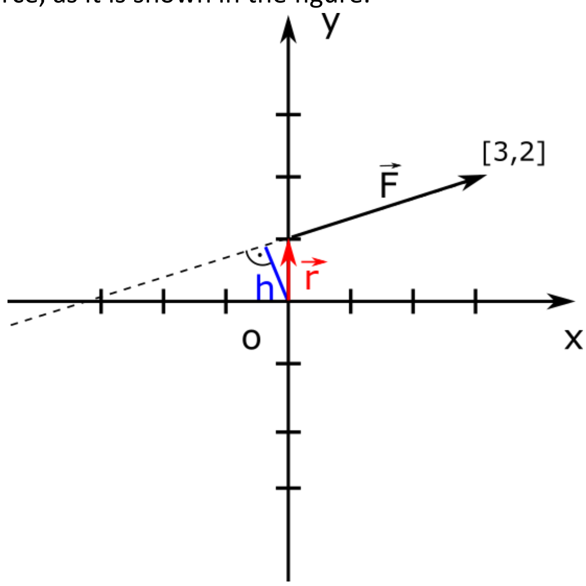


This new vector will be called as r. Additionally we know description of these vector, and vector F = 3i+1j+0k, and vector r=0i+1j+0k. Based on this information we can calculate moment from the definition that

$$\vec{M}_o = \vec{r} \times \vec{F}$$

$$\vec{M}_o = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 3 & 1 & 0 \end{vmatrix} (0-0)\hat{\imath} + (0-0)\hat{\jmath} + (0-3)\hat{k} = -3\hat{k}$$

Usually we are more interested in the value of the moment than how its placed in the space. This is why it is better to calculate moment by finding distance which will be perpendicular to the direction of the force, as it is shown in the figure.



In order to calculate value of the vector product we need to know what are the values of each of vectors and the sine of the angle between these vectors.

$$|\vec{M}_o| = |\vec{r}| * |\vec{F}| * \sin(\vec{r}, \vec{F})$$

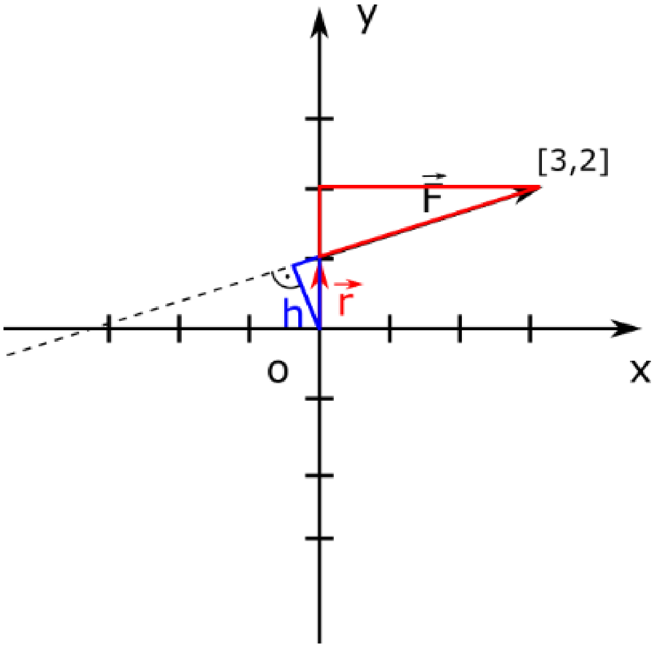
When instead of “r” we will take “h” then the angle between h and F will be 90 degrees, it means that sine of this angle is equal to 1. Finally we are obtaining equation in a form:

$$|\vec{M}_o| = h * \overrightarrow{|F|}$$

The only thing that left is to find values of F and h.

$$\overrightarrow{|F|} = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$$

Now when we look closely to the drawing we can see that highlighted triangles (blue and red) are similar to each other. Based on this we can calculate value of h.



$$\frac{1}{h} = \frac{|\vec{F}|}{3} \rightarrow h = \frac{3}{|\vec{F}|} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

now, we can calculate value of the moment

$$|\vec{M}_o| = h * \overrightarrow{|F|} = \frac{3\sqrt{10}}{10} * \sqrt{10} = \frac{30}{10} = 3$$

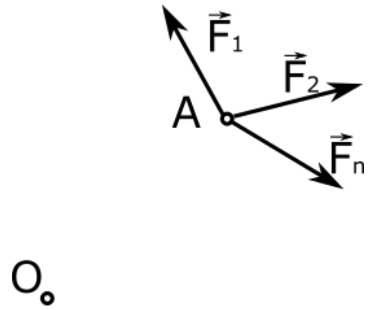
As you can see in both ways we obtain the same result. Vector of the moment is equal to 3. However, in first type of solution we are ale able to determine sense of obtained vector. You can notice that we obtained vector with negative sign next to it. It means that sense of our vector is opposite to the positive value of axis Z. In this case vector of the moment will be pointing into the figure.

Varignon's theorem

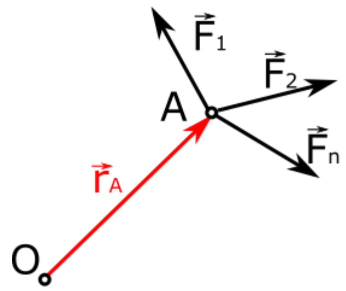
The moment of the resultant force of forces at any point is equal to the sum of the moments of the component forces relative to the same point.

This theorem might seems to be complicated, but actually it is very simple.

Let's assume that we have many different forces which are acting in the way given in the figure. We can see that directions of these forces are going through the point A, but we want to find moment form these forces relative to point O.



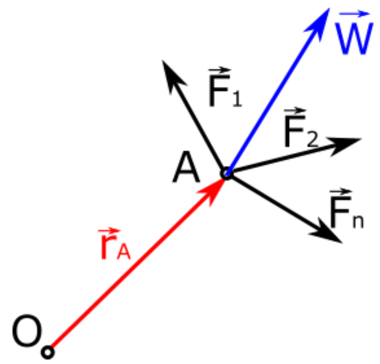
1. First we need to introduce vector of "arm".



2. In this way we can calculate moments from each of these forces and then obtain the whole moment.

$$\sum_{i=1}^{n} \vec{M}_o(\vec{F_i}) = \vec{r_A} \times \vec{F_1} + \vec{r_A} \times \vec{F_2} + \cdots + \vec{r_A} \times \vec{F_n} = \sum_{i=1}^{n} \vec{r_A} \times \vec{F_i}$$

3. But we know that we can reduce system of forces to one resultant force W.



$$\vec{W} = \sum_{i=1}^{n} \vec{F_i}$$

$$\vec{M}_o(\vec{F}) = \vec{r_A} \times \vec{W}$$

Based on this information we can write

$$\sum_{i=1}^{n} \vec{M}_o(\vec{F_i}) = \vec{r_A} \times \sum_{i=1}^{n} \vec{F_i} = \vec{r_A} \times \vec{W}$$

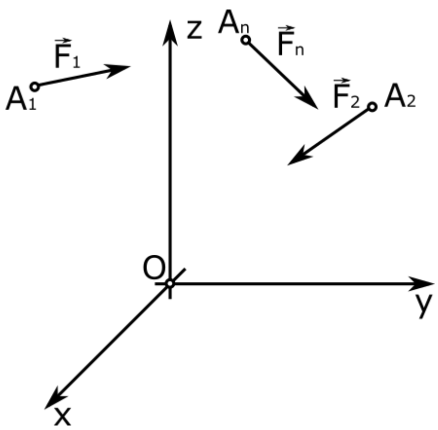
Finally we can say that instead of calculating moment from each of force relative to some point it is enough to calculate moment from the resultant force of these forces relative to the same point.

The theorem has been proved on the basis of a convergent system of forces, but it is of a general nature and applies to any system of forces that has a resultant force.

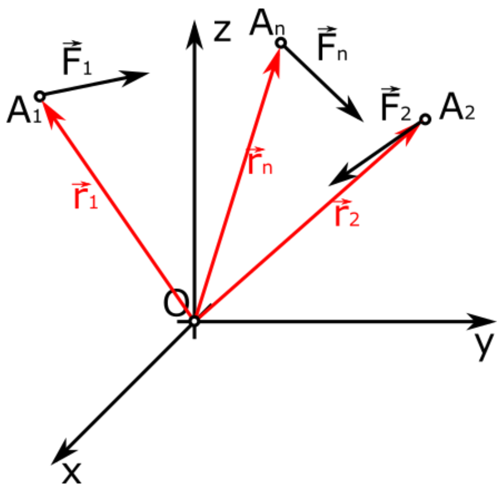
Reduction of the Divergent system of forces

We speak of a divergent system of forces when the directions of forces are distributed in space freely.

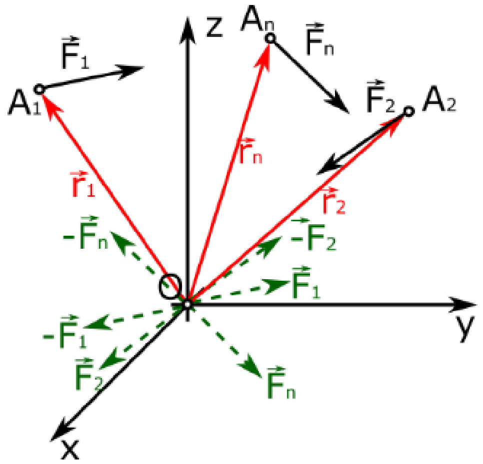
1. Let's say that we have divergent system of forces. Point O will be our reduction point relative to which we will reduce this system of forces.



2. Let's introduce vectors r, which are determining position of each force relative to point O.

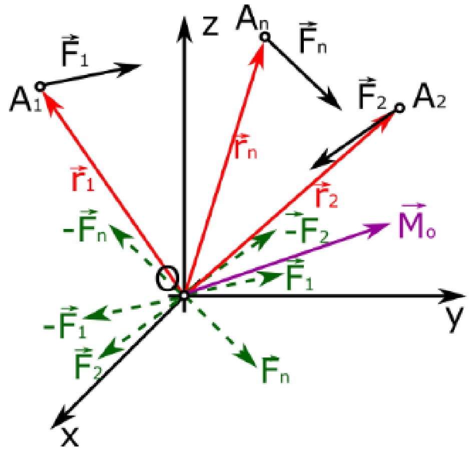


3. Next at point O we will introduce pairs of vectors (zero pairs) in, a way that do not have influence on the system.



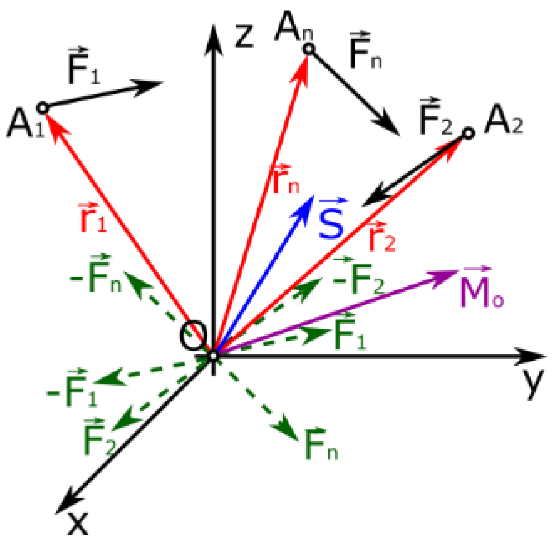
Right now we obtained two things. First we can see that we have Couples of forces, and second we have convergent system of forces from the forces that left after creating Couples of forces.

4. We know that Couple of forces is creating moment around a specific point. In our case each of Couples is creating moment relative to point O. We can reduce these moments to one moment MO relative to point O.



Moment M_O we will call as a Main Moment of the system.

5. After reduction of Couples of forces to Main Moment only convergent system of forces left. We already know that each convergent system might be reduced to one force. In this case we will reduce this system to force S.



Force S we will call as a Main Force of the system.

$$\vec{S} = \sum_{i=1}^{n} \vec{F_i}$$

$$\vec{M}_o(\vec{F_i}) = \vec{r_i} \times \vec{F_i}$$

$$\vec{M}_o = \sum_{i=1}^{n} \vec{M}_o(\vec{F_i}) = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}$$

Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one S force attached in selected reduction point and a pair of forces with the moment M_O.

In the case of a rectangular coordinate system

$$\vec{S} = S_x\hat{\imath} + S_y\hat{\jmath} + S_z\hat{k}$$

$$\left.\begin{aligned} S_x &= \sum_{i=1}^{n} F_{xi} \\ S_y &= \sum_{i=1}^{n} F_{yi} \\ S_z &= \sum_{i=1}^{n} F_{zi} \end{aligned}\right\}$$

$$\vec{M}_o = M_{Ox}\hat{\imath} + M_{Oy}\hat{\jmath} + M_{Oz}\hat{k}$$

$$\left.\begin{aligned} M_{Ox} &= \sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} y_i F_{zi} - z_i F_{yi} \\ M_{Oy} &= \sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} z_i F_{xi} - x_i F_{zi} \\ M_{Oz} &= \sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} x_i F_{yi} - y_i F_{xi} \end{aligned}\right\}$$

For any divergent system of forces to be in balance, the following conditions must be met

$$\vec{S} = 0$$
$$\vec{M}_o = 0$$

It means, that the divergent system of forces will be in the balance if the sum of these forces and their moment relative to any point will be equal to zero.

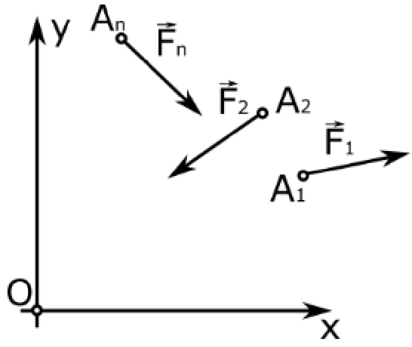
$$\sum_{i=1}^{n} \vec{F_i} = 0$$
$$\sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} = 0$$

The divergent system of forces will be in the balance if the sum of projections of all forces on axises of coordinates system and sum of the moments of all forces relative to the axises of coordinate system will be equal to zero.

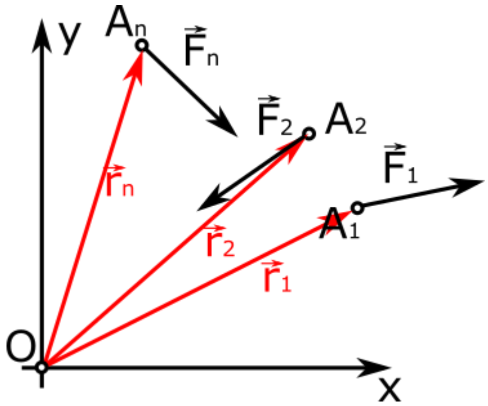
$$\left.\begin{matrix} \sum_{i=1}^{n} F_{xi} = 0; & \sum_{i=1}^{n} F_{yi} = 0; & \sum_{i=1}^{n} F_{zi} = 0 \\ \sum_{i=1}^{n} M_{ix} = 0; & \sum_{i=1}^{n} M_{iy} = 0; & \sum_{i=1}^{n} M_{iz} = 0 \end{matrix}\right\}$$

Planar system of divergent forces

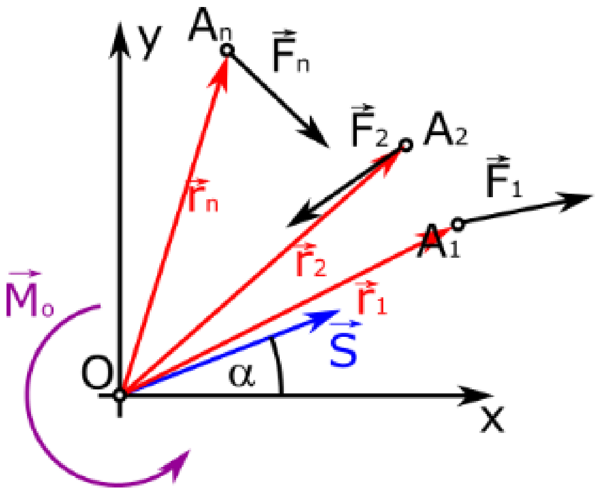
Now let’s consider planar system of divergent forces.



1. First we will do the same thing that we did for 3D, so let's introduce vectors r, which are determining position of each force relative to point O.



2. In case of planar system of force we can reduce such a system to one resultant force S which can be attached in any point we want, and to Couple of forces with the moment M_O.



$$\vec{S} = \sum_{i=1}^{n} \vec{F_i}; \quad \vec{M}_o = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}$$

$$\vec{S} = S_x \hat{\imath} + S_y \hat{\jmath}$$

$$\left. \begin{aligned} S_x &= \sum_{i=1}^{n} F_{xi} = \sum_{i=1}^{n} F_i \cos(\alpha_i) \\ S_y &= \sum_{i=1}^{n} F_{yi} = \sum_{i=1}^{n} F_i \sin(\alpha_i) \end{aligned} \right\}$$

$$|\vec{S}| = \sqrt{{S_x}^2 + {S_y}^2}$$

$$\tan \alpha = \frac{S_y}{S_x}$$

$y = \frac{S_y}{S_x} * x - \frac{M_O}{S_x} -$ the equation of the straight along which the resultant force works

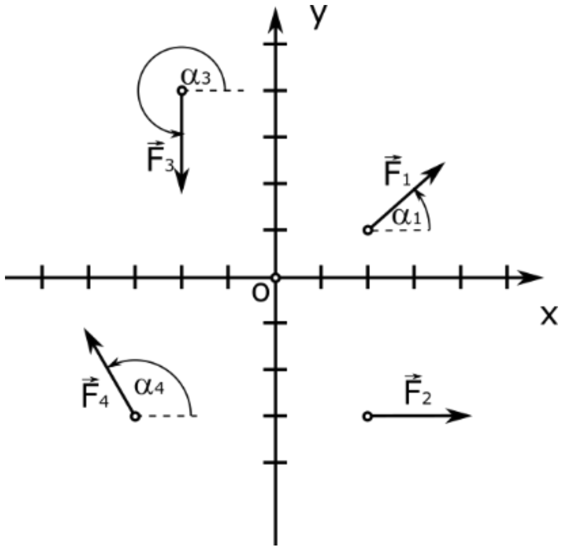
$r = \frac{M_O}{S} -$ distance of the direction of the resultant force from the center of reduction

Example 2.

Make a reduction of a given system of forces and define the resultant force. Data: F_1=200N, F_2=600N, F_3=300N, F_4=500N; points where forces are attached: $A_1(2,1)$, $A_2(2,-4)$, $A_3(-2,4)$, $A_4(-3,-3)$; angles between positive part of axis X and direction of force:

$\alpha_1=45^\circ$, $\alpha_2=0^\circ$, $\alpha_3=270^\circ$, $\alpha_4=120^\circ$,

1. First of all we need to draw everything in a coordinate system.



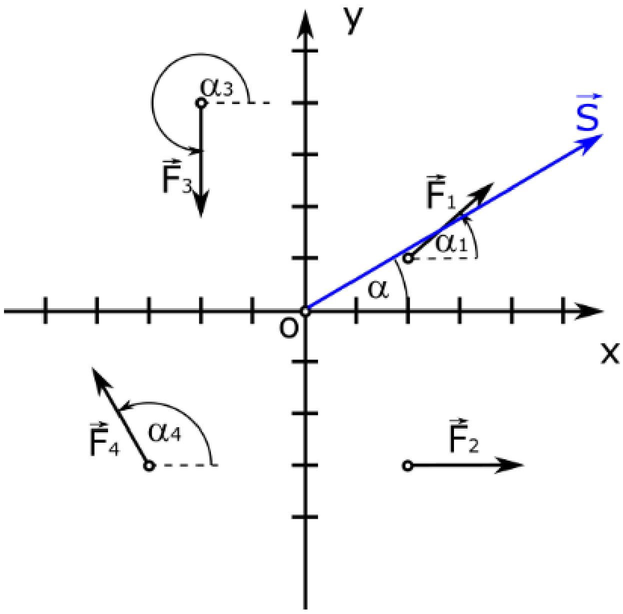
2. Next based on the equation given above we can find main force S of this system of forces.

$S_x = \sum_{i=1}^{n} F_i \cos(\alpha_i)$ $=F_1\cos(\alpha_1) + F_2\cos(\alpha_2) + F_3\cos(\alpha_3) + F_4\cos(\alpha_4) = 491N$

$S_y = \sum_{i=1}^{n} F_i \sin(\alpha_i)$$=F_1\sin(\alpha_1) + F_2\sin(\alpha_2) + F_3\sin(\alpha_3) + F_4\sin(\alpha_4) = 273{,}5N$

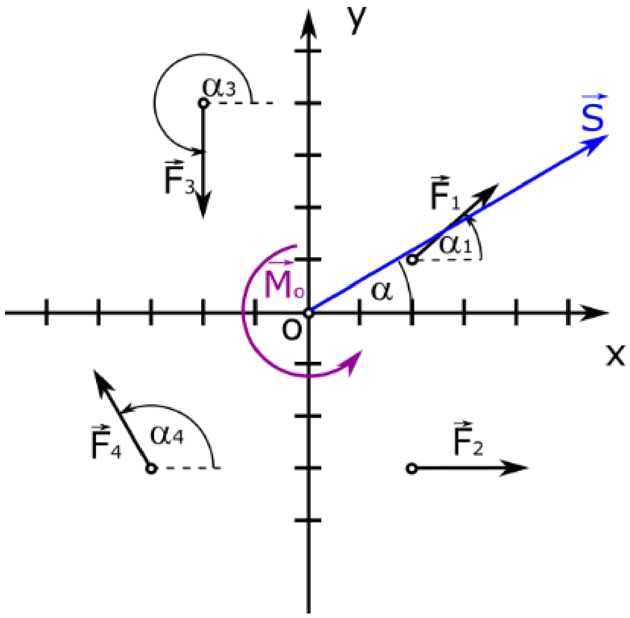
$$|\vec{S}| = \sqrt{{S_x}^2 + {S_y}^2} = 562N$$

$$\sin\alpha = \frac{S_y}{S} = 0{,}487 \rightarrow \alpha = 29°07'$$



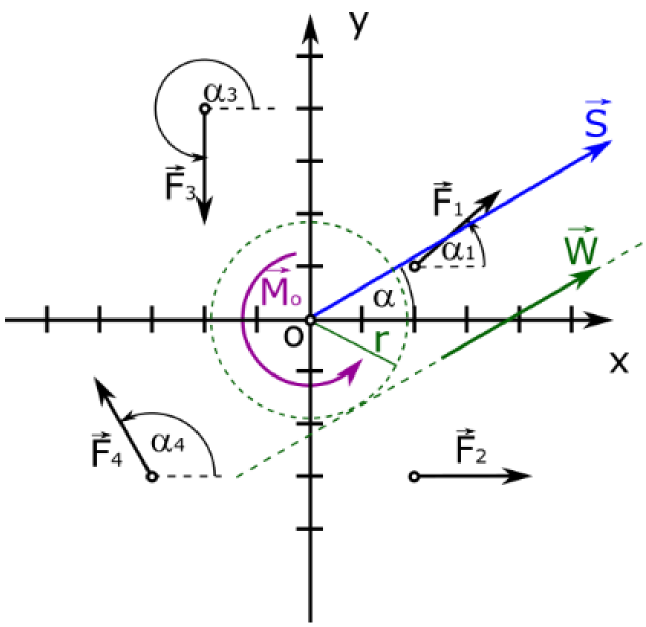
3. After that we can find main moment of the system M_O. Due to the fact that this system is a planar system we just need to find moment rotating relating to the axis Z.

$$M_{Oz} = \sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} x_i F_{yi} - y_i F_{xi} = \sum_{i=1}^{n} x_i F_i \cos(\alpha_i) - y_i F_i \sin(\alpha_i)$$
$$= F_1 \sin(\alpha_1) * 2 - F_1 \cos(\alpha_1) * 1 + F_2 \sin(\alpha_2) * 2 - F_2 \cos(\alpha_2) * (-4) + F_3 \sin(\alpha_3) * (-2) - F_3 \cos(\alpha_3) * 4 + F_4 \sin(\alpha_4) * (-3) - F_4 \cos(\alpha_4) * (-3) = 1092Nm$$



4. Next we can find equation of the straight along which the resultant force works, and the distance of the direction of the resultant force from the center of reduction.

$$y = \frac{S_y}{S_x} * x - \frac{M_O}{S_x} = \frac{273,5}{491} x - \frac{1092}{491} \rightarrow y = 0,557x - 2,22$$
$$r = \frac{M_O}{S} = \frac{1092}{562} = 1,9m$$



Here are three ceases when planar divergent system of forces will be in equilibrium.

I. The sums of all projections of the force on the two axes of the coordinate system and the sum of the moments of these forces relative to any point in the plane of force must be equal to zero.

$$\sum_{i=1}^{n} F_{xi} = 0; \quad \sum_{i=1}^{n} F_{yi} = 0; \quad \sum_{i=1}^{n} M_O = 0$$

II. The sum of moments relative to two points on the plane must be equal to zero, and the sum of projections on any axis not perpendicular to the segment connecting the points on which the moments were calculated must be equal to zero.

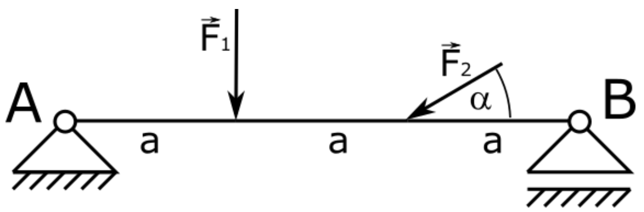
$$\sum_{i=1}^{n} F_{xi} = 0; \quad \sum_{i=1}^{n} M_A = 0; \quad \sum_{i=1}^{n} M_B = 0$$

III. The sum of moments relative to any three points lying on the same plane, but not lying on one straight line must be equal to zero.

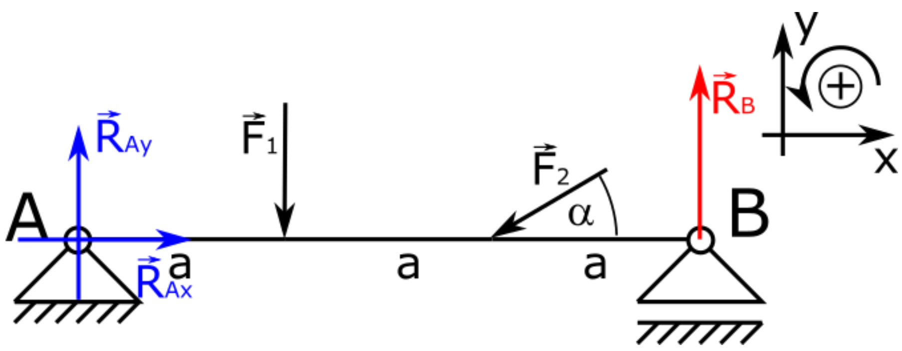
$$\sum_{i=1}^{n} M_A = 0; \quad \sum_{i=1}^{n} M_B = 0; \quad \sum_{i=1}^{n} M_C = 0$$

Example 3.

Find reactions in supports. Data: F_1=60N, F_2=120N, α=30°, a=2m.

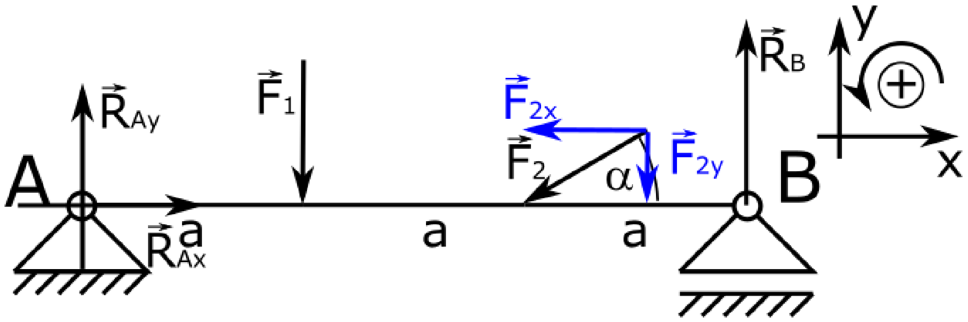


1. First of all we need to ask ourselves what type of forces system do we have here. It is obvious that this system is not convergent, and we cannot use theorem of three forces to solve this example. We have two outer forces F1 and F2, and three components of supports forces R_{Ax}, R_{Ay} and R_B. There are three unknown forces that we need to find, and which were marked in the figure. Additionally we need to introduce coordinate system according to which we will be projecting forces, and also we need to assume in which way moment will be rotating in positive way. As you can see it is assume that moments rotating anticlockwise will be with positive sign.



2. We have three unknowns and we know that this system is in equilibrium state. Additionally this is planar system, so we can write three equations of equilibrium. Projection of forces over axis X, projection of forces over axis Y and sum of moments relative to point. We can pick any point we want, but later I will write which point we will choose and why.

$$\sum_{i=1}^{n} F_{xi} = 0; \quad \sum_{i=1}^{n} F_{yi} = 0; \quad \sum_{i=1}^{n} M_O = 0$$



Let's start with projections of forces;

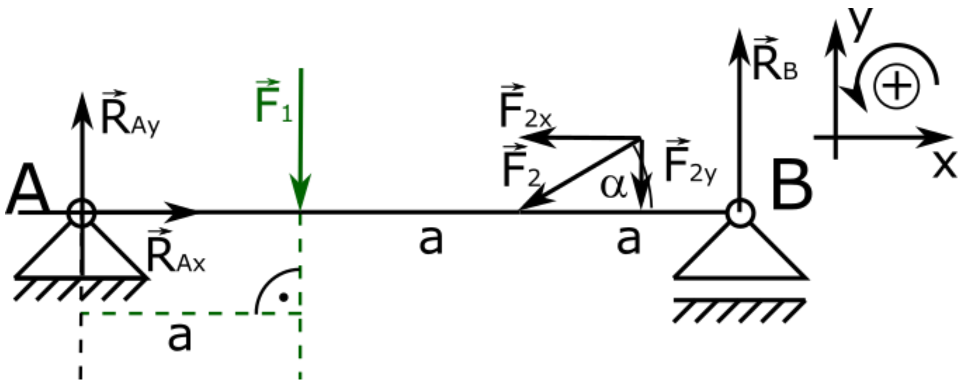
$$\sum_{i=1}^{n} F_{xi} = 0 = R_{Ax} - F_{2x} = R_{Ax} - F_2 \cos\alpha$$

$$\sum_{i=1}^{n} F_{yi} = 0 = R_{Ay} - F_1 - F_{2y} + R_B = R_{Ay} - F_1 - F_2 \sin\alpha + R_B$$

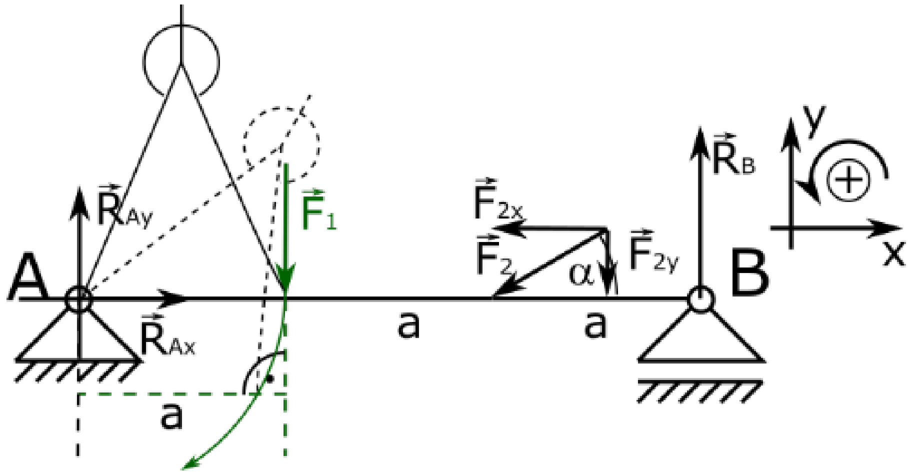
Now we have two equations of balance for this system, but we have three unknowns, so we will need one more equations.

3. Third equations of balance for this system will a sum of the moments relative to point A. It was written above that one can choose any point. But it is wise to choose point through which will be going most directions of forces, because then moments from these forces will be equal to zero. What is also important it is good to choose a point where we can reduce most unknowns. If you look closely to the figure you can see that through point A are going two directions of unknown forces it means that moments from these forces will be equal to zero so we would not include these force in our equation of moments.

I will create equation of the moments step by step. We are starting at point A and moving towards point B. At point A we have two forces R_{Ay} and R_{Ax}, but as I mentioned before, there will be no moments from this forces because directions of these are going through the point A. Next we can see force F_1. As it was written before, the best way to find what moment we have from any force, is to find segment between direction of our force and relative point which will be perpendicular to that direction. This segment is marked in the figure with green color, and has length equal to “a”.



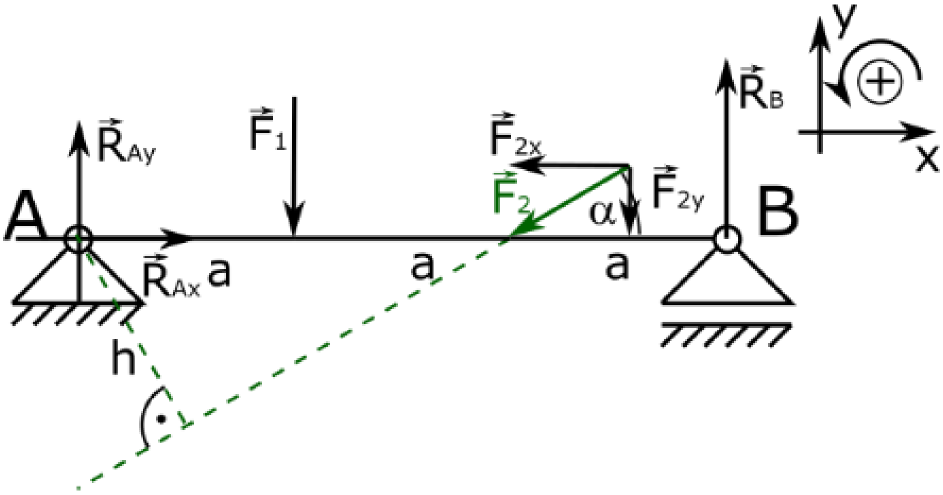
After finding distance you need to determine if the moment from this force will be with positive or negative sign. The best way is to imagine yourself that you have compass, and one end of a compass you are attaching to the point around which force is trying to rotate, and second end you are trying to rotate according to the sense of the force.



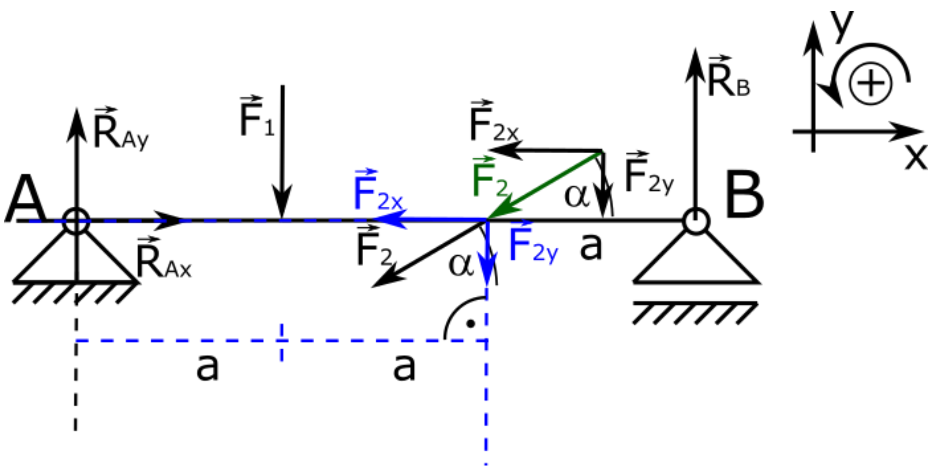
As you can see force F_1 is trying to rotate around point A clockwise. Moment from this force will be negative, because we assume at the beginning that positive sign of moment will be for anticlockwise rotation. Now we can include moment from this force in our equation.

$$\sum_{i=1}^{n} M_A = 0 = -F_1 * a$$

4. Next force that we can see on our system is force F_2. If we do the same steps as we did for previous force distance from the direction of this force to the point A will be looking as it is shown in figure. Sometimes it might be hard to find value of “h”.



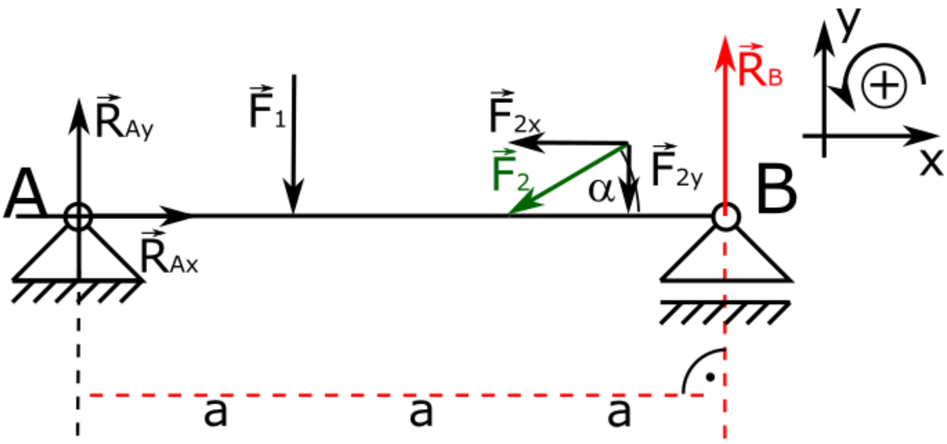
This is why sometimes it is better to split force to its components as it is shown in this figure. We have two components F_{2x} and F_{2y}. One can notice that direction of force F_{2x} is going through the point A, so there will be no moment from this component of force F_2. The only moment will be from the F_{2y} component, and the segment between direction of this component and point A is equal to “2a”. Now we just need to determine sign of the moment from this component. It is clear that the sign will be the same as for the previous force, because sense of both forces are the same.



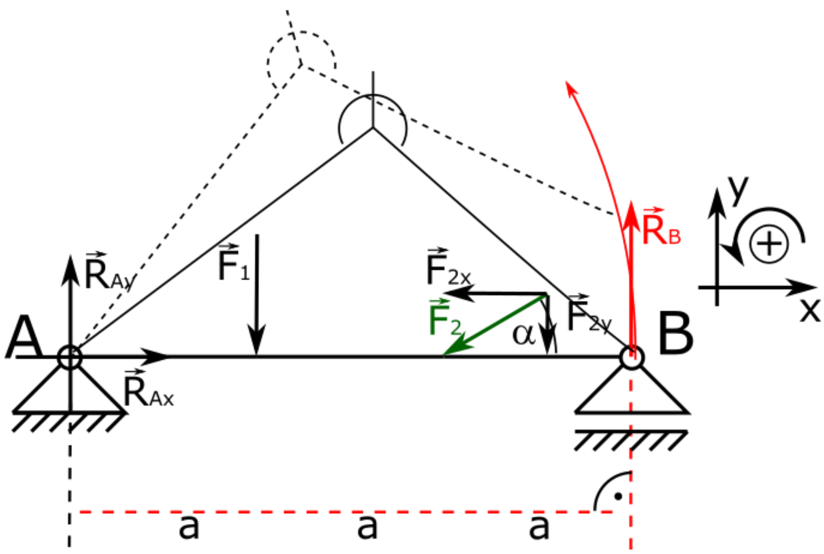
Now the equation of moment for this part will look like this:

$$\sum_{i=1}^{n} M_A = 0 = -F_1 * a - F_2 \sin\alpha * 2a$$

5. Finally at the end we have unknown force R_B. For this force we need to do the same thing that we did for previous two forces. The segment between direction of this force and point A is equal to “3a”, what can be clearly seen. The only thing left is to determine sing for the moment from this force. Let’s do the same as we did for first force, so we will use our imaginary compass.



The only thing left is to determine sign for the moment from this force. Let’s do the same as we did for first force, so we will use our imaginary compass. As you can see force R_B is trying to rotate around point A anticlockwise. Moment from this force will be positive, because we assume at the beginning that positive sign of moment will be for anticlockwise rotation. Now we can include moment from this force in our equation.



Final equation of the moments for this example will has such form:

$$\sum_{i=1}^{n} M_A = 0 = -F_1 * a - F_2 \sin\alpha * 2a + R_B * 3a$$

Finally we have three equations and three unknows:

$$\sum_{i=1}^{n} F_{xi} = 0 = R_{Ax} - F_{2x} = R_{Ax} - F_2 \cos\alpha$$

$$\sum_{i=1}^{n} F_{yi} = 0 = R_{Ay} - F_1 - F_{2y} + R_B = R_{Ay} - F_1 - F_2 \sin\alpha + R_B$$

$$\sum_{i=1}^{n} M_A = 0 = -F_1 * a - F_2 \sin\alpha * 2a + R_B * 3a$$

and the final solution:

$$R_{Ax} - F_2 \cos\alpha = 0 \rightarrow R_{Ax} = F_2 \cos\alpha = 120 * \frac{\sqrt{3}}{2} = \underline{60\sqrt{3}N}$$

$$-F_1 * a - F_2 \sin\alpha * 2a + R_B * 3a = 0 \rightarrow R_B = \frac{F_1 * a + F_2 \sin\alpha * 2a}{3a} = \frac{60 + 120 * \frac{1}{2} * 2}{3} = \underline{60N}$$

$$R_{Ay} - F_1 - F_2 \sin\alpha + R_B = 0 \rightarrow R_{Ay} = F_1 + F_2 \sin\alpha - R_B = 60 + 120 * \frac{1}{2} - 60 = \underline{60N}$$