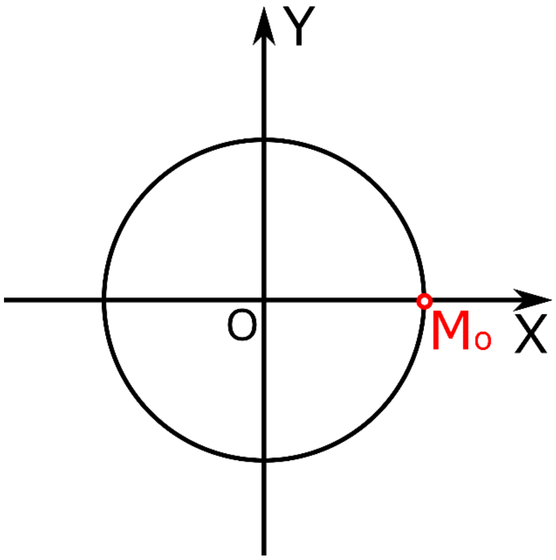
Motion of a point on the circle

The previous topic concerned the motion of a point along a curvilinear path. However, the present topic is a special kind of curvilinear motion, where the radius of curvature ρ is constant throughout the motion and equals the radial r of the circle.

Suppose point M moves on a circle with radius r. The beginning of the motion is on the positive side of the X axis (Mo) and changes along with the angle ϕ. Thus, point M travels the arc MoM.



We can calculate the length of the arc according to the equation.

$$s = r * \phi$$

We know from the previous topic that velocity is equal

$$V = \frac{ds}{dt}$$

Since the radius is constant with time, the only value that changes with time is the angle. Therefore we can write.

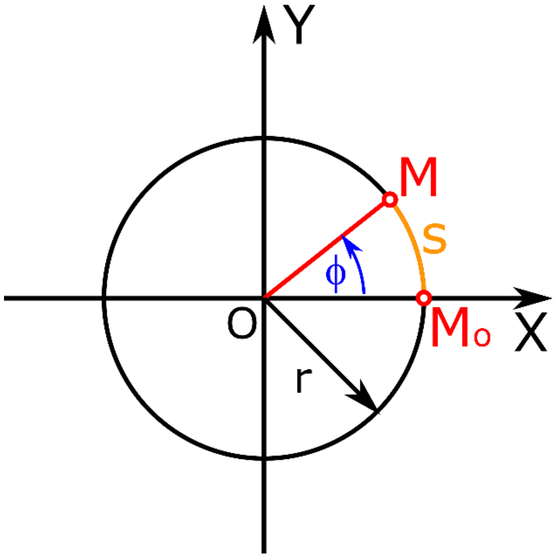
$$V = \frac{ds}{dt} = r\frac{d\phi}{dt}$$

The derivative of an angle with time is equal to the angular velocity.

$$\frac{d\phi}{dt} = \omega$$

finally, we have

$$V = \frac{ds}{dt} = r\frac{d\phi}{dt} = r * \omega$$



We already know how to calculate the velocity in a circular motion. In the next step, we will find how to determine acceleration of the point.

From the previous topic, we know that in the case of a curvilinear motion we distinguish two components of acceleration, i.e. tangential acceleration and normal acceleration.

$$\vec{a} = \vec{a}_\tau + \vec{a}_n$$

In general, we can write these accelerations as

$$a_n = \frac{V^2}{\rho}$$

$$a_\tau = \frac{dV}{dt}$$

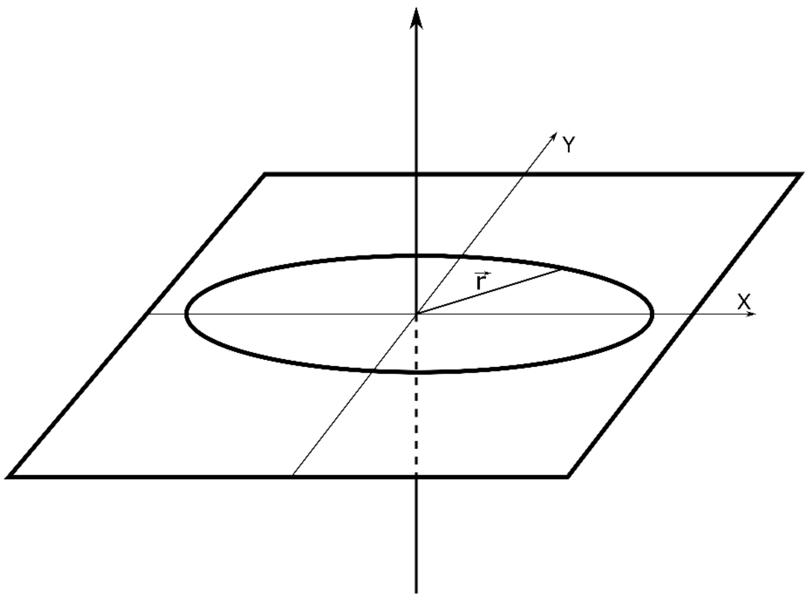
Since the radius of curvature is constant, we transform the above equations to the following form.

$$a_\tau = \frac{dV}{dt} = r\frac{d\omega}{dt} = r\frac{d^2\phi}{dt^2} = r * \varepsilon$$

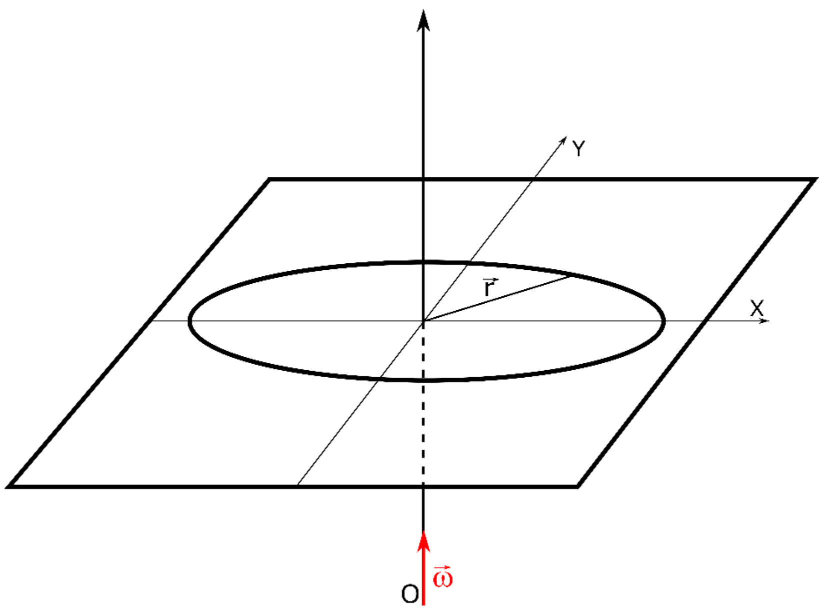
$$a_n = \frac{V^2}{r} = \frac{(r * \omega)^2}{r} = \frac{r^2\omega^2}{r} = r * \omega^2$$

In the following part of the discussion, let we define what ω is for us.

ω is a vector quantity. Suppose we have a point that moves in a certain plane in a circular motion. The radius of the circle is r.



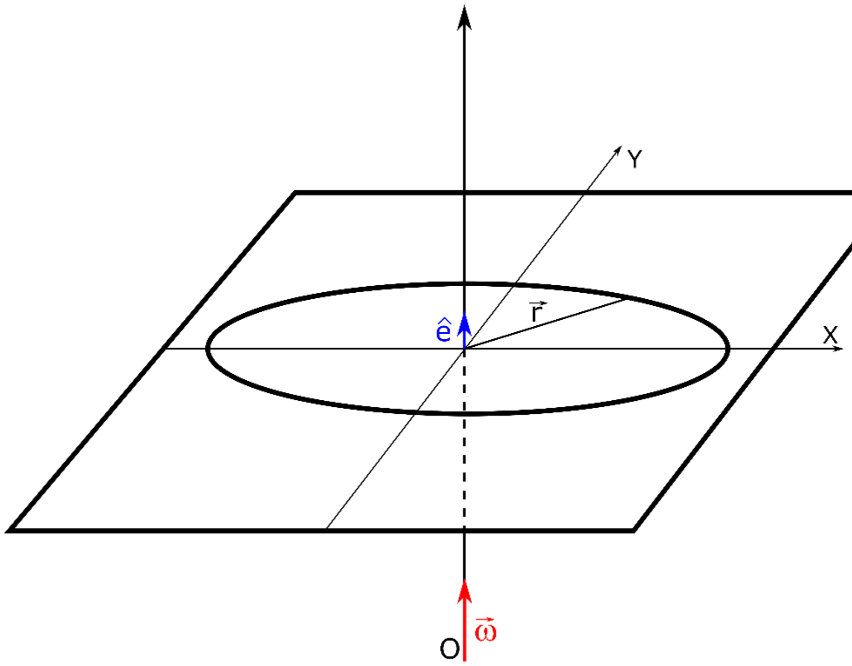
The angular velocity vector will lie on the axis perpendicular to the plane on which the path of the point is. This axis is also the axis of rotation of our point. Moreover, we can move freely this vector along the line of action.



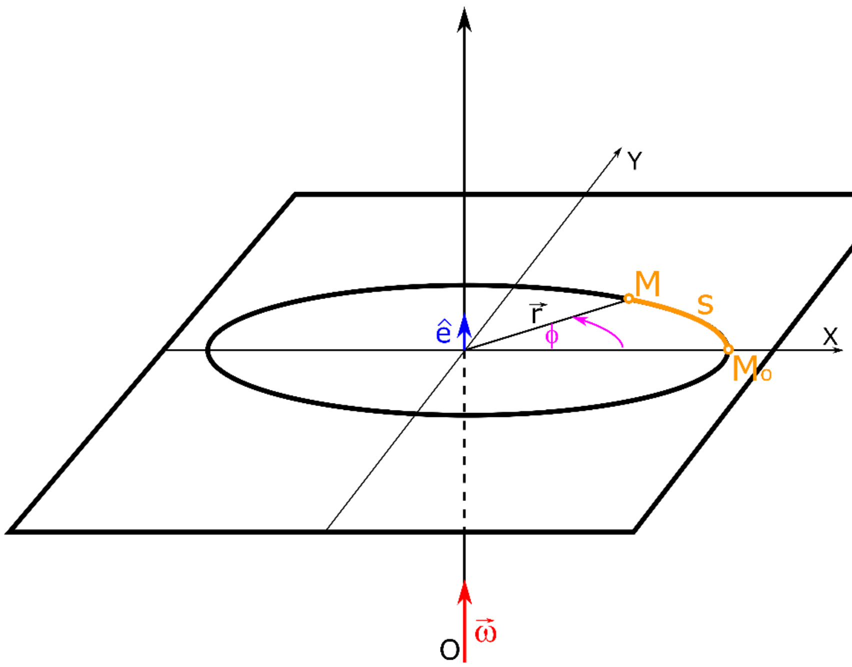
The angular velocity vector $\vec{\omega}$ can be written as follows

$$\vec{\omega} = \omega\hat{e} = \frac{d\phi}{dt}\hat{e}$$

where $\hat{e}$ will be a unit vector for us in the direction of the Z axis.



Suppose point M moves on a circle with radius r. The beginning of the motion at point Mo and changes with the angle ϕ. Thus, point M travels the arc s (MoM).



Since the point is moving, it means that it has a certain linear velocity $\vec{V}$. We know that the linear velocity is a vector quantity and as noted earlier, it depends on the radius $\vec{R}$ and the angular velocity $\vec{\omega}$. Hence, by definition, we can write that

$$\vec{V} = \vec{\omega} \times \vec{R}$$

To get the value of the velocity vector, we have to use the following formula.

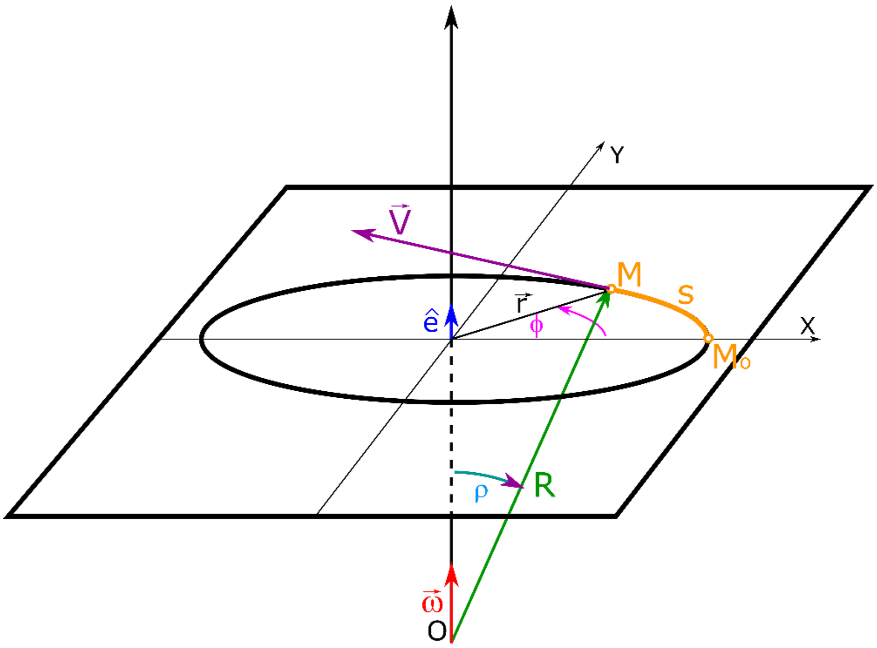
$$|\vec{V}| = \omega * R * \sin \rho$$

We can see clearly that in this way we get the expression in following form,

$$|\vec{V}| = \omega * R * \sin \rho = \omega * r$$

where the vectors are perpendicular to each other.

$$\vec{V} \perp \vec{\omega} \perp \vec{r}$$



With the above knowledge of velocity in the circular motion, we can move to acceleration.

We know acceleration is derived from velocity. We can write the following equation

$$\vec{a} = \frac{d\vec{V}}{dt}$$

due to the fact that

$$\vec{V} = \vec{\omega} \times \vec{R}$$

let's write the above equations as follows

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{R}) = \frac{d\vec{\omega}}{dt} \times \vec{R} + \vec{\omega} \times \frac{d\vec{R}}{dt}$$

It is clear that we have obtained the derivative of the angular velocity over time. I know from the previous considerations that the derivative of speed over time is equal to acceleration. In this case we are talking about angular velocity, therefore the resulting acceleration will be angular acceleration.

$$\frac{d\vec{\omega}}{dt} = \vec{\varepsilon}$$

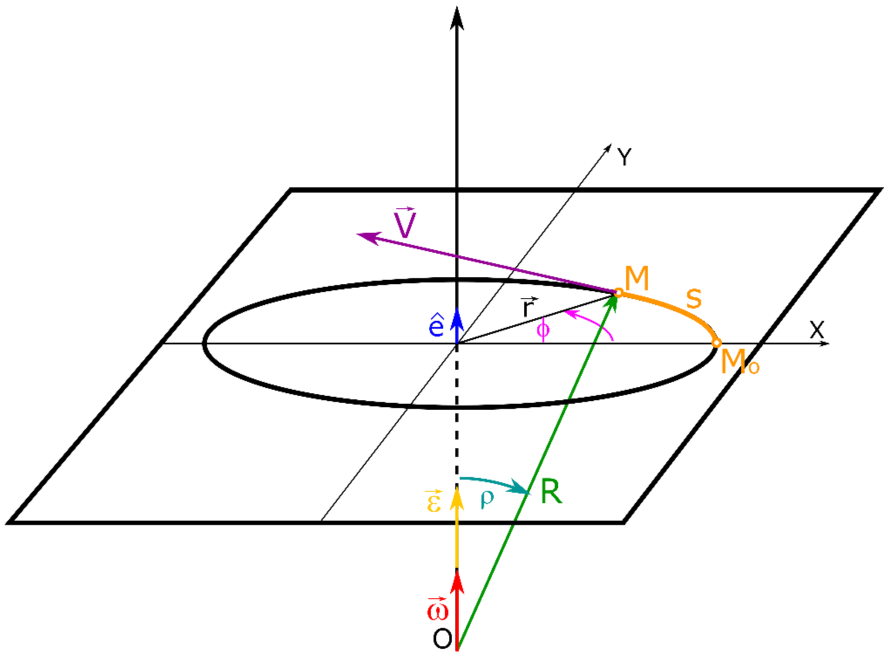
Additionally, we obtained the derivative of the length vector over time, which is actually the derivative of the path change over time. We know that this derivative is going to be equal to the linear velocity.

$$\frac{d\vec{R}}{dt} = \vec{V}$$

Taking the above into account, we can say that the vector of linear acceleration in a circular motion has the following form.

$$\vec{a} = \vec{\varepsilon} \times \vec{R} + \vec{\omega} \times \vec{V}$$

Since the angular velocity vector $\vec{\omega}$ is in the direction of the Z axis and the angular acceleration vector $\vec{\varepsilon}$ is a derivative of the angular velocity vector over time, the angular acceleration vector must also be in the direction of the angular velocity.



in the equation

$$\vec{a} = \vec{\varepsilon} \times \vec{R} + \vec{\omega} \times \vec{V}$$

the first term corresponds to the tangential acceleration

$$\vec{a}_\tau = \vec{\varepsilon} \times \vec{R}$$

$$|\vec{a}_\tau| = \left|\vec{\varepsilon} \times \vec{R}\right| = \varepsilon * R * \sin\rho = \varepsilon * r$$

while the second term corresponds to normal acceleration.

$$\vec{a}_n = \vec{\omega} \times \vec{V}$$

$$|\vec{a}_n| = |\vec{\omega} \times \vec{V}| = \omega * V * \sin\frac{\pi}{2} = \omega^2 * r = \frac{V^2}{r}$$

The normal acceleration vector $\vec{a}_n$ is perpendicular to the vectors $\vec{\omega}$ and $\vec{V}$ and it sense is to the center of the circle.

The angular acceleration may be

$$\vec{\varepsilon} > 0; \ \vec{\varepsilon} < 0$$

or

$$\vec{\varepsilon} = 0 \ then \ \vec{V} = const$$

at start-up

$$\vec{V} = 0 \ and \ \vec{\varepsilon} \neq 0$$

Example 1.

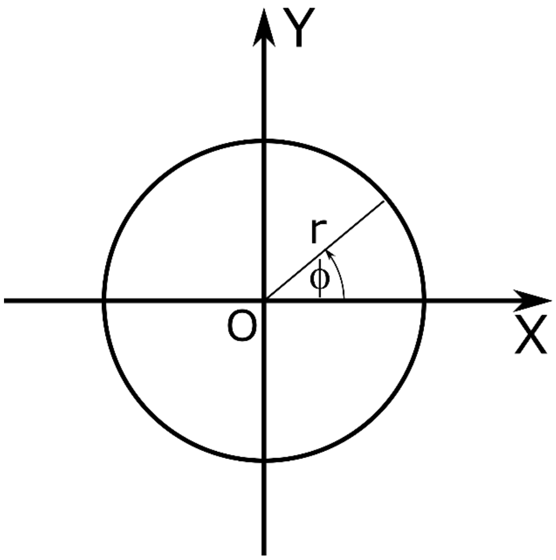
The point moves on a circle with a radius r = 1,5m according to the equation

$$s = 0{,}5 + 0{,}06t + 0{,}1t^3$$

Find the angular and linear velocity and acceleration of a point for time t_1 = 5s.

First, as always, we start with the drawing. So let's draw the path of the point in the coordinate system. The center of the circle will be in the center of the coordinate system.

Suppose point M starts on the X axis and moves according to the marked change in angle.



At the beginning it will be good to find out where the point is at any given moment.

By substituting t_1 into the given equation, we can determine what path has been traveled since the beginning of the movement.

$$s(t_1) = 0{,}5 + 0{,}06 * 5 + 0{,}1 * 5^3 = 13{,}3m$$

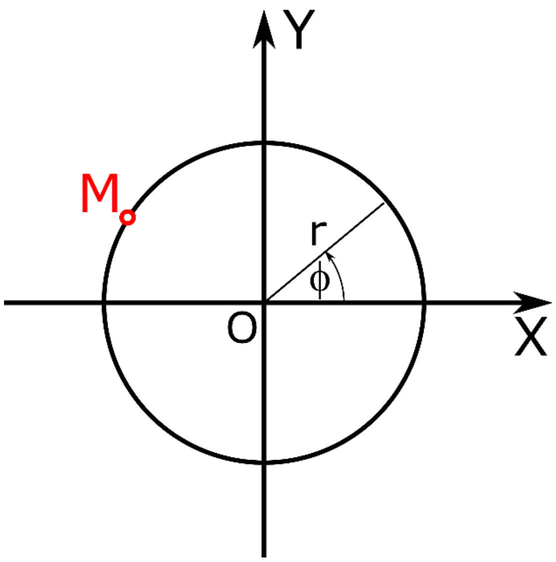
Once we know the value by which the point will move and the value of the radius, we can determine how the angle has changed with time.

$$\phi = \frac{s}{r} = \frac{13,3}{1,5} = 8,86\,rad$$

It remains to convert radians to degrees

$$\phi = 508°$$

So our point covered one full turn and less than a half of the turn.



Next, we will determine the velocities. We know the derivative of the path will be equal to the linear velocity.

$$V = \frac{ds}{dt}$$

$$V = \frac{d(0,5 + 0,06t + 0,1t^3)}{dt} = 0,06 + 0,3t^2$$

$$V(t_1) = 0,06 + 0,3 * 5^2 = 7,56\,m/s$$

additionally, we know that

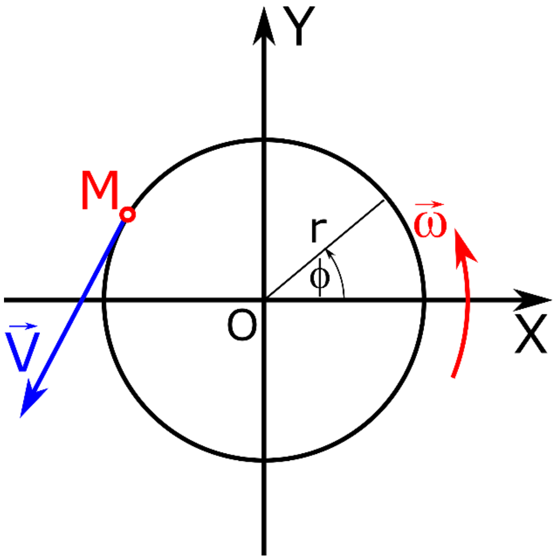
$$V = \omega * r$$

where, after the transformation we have

$$\omega = \frac{V}{r} = \frac{0,06 + 0,3t^2}{1,5}$$

$$\omega(t_1) = \frac{0,06 + 0,3 * 5^2}{1,5} = 5,04$$

Knowing the velocity values, we can now put them on the drawing. Since we have assumed that the point is moving counterclockwise and the angular velocity $\vec{\omega}$ value has a positive sign, it must also rotate counterclockwise. Additionally, using the rule of a right-hand screw, we can determine that the sense of the angular velocity vector $\vec{\omega}$ will be towards us. The linear velocity $\vec{V}$ shall be drawn on the tangent to the path and the sense of this velocity must be as the rotation of the angular velocity.



After the velocities, we can move to accelerations.

Taking into account the previous information, we know that

$$a_\tau = \frac{dV}{dt}$$

$$a_\tau = \frac{d(0{,}06 + 0{,}3t^2)}{dt} = 0{,}6t$$

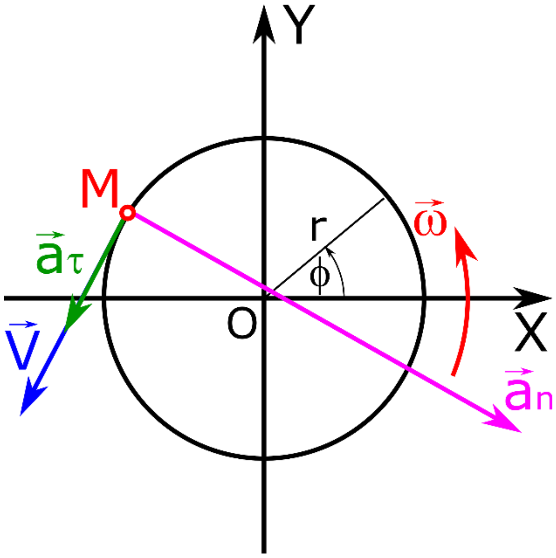
$$a_\tau(t_1) = 0{,}6 * 5 = 3$$

and

$$a_n = \frac{V^2}{r}$$

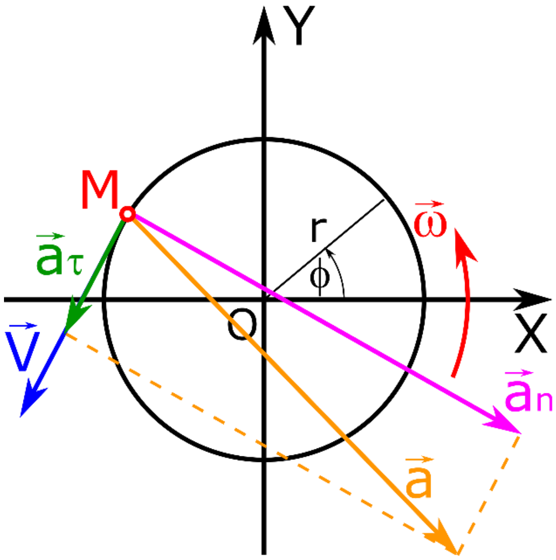
$$a_n(t_1) = \frac{7{,}56^2}{1{,}5} = 38{,}1$$

Knowing the values of the acceleration components, we can put them on our drawing. The tangential acceleration will be in the direction of linear velocity, and its sense will be same as sense of linear velocity, because we have a positive value. Whereas the normal acceleration will be in the direction perpendicular to the direction of the tangential acceleration and its sense will always be in the direction of the axis of rotation.



In the next step, we can determine the linear acceleration of the point

$$a = \sqrt{a_\tau^2 + a_n^2} = \sqrt{3^2 + 38{,}1^2} = 38{,}22$$

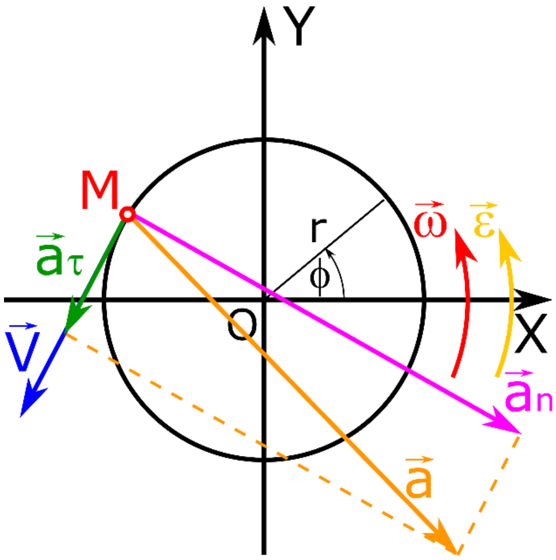


Finally, it remains to determine the angular acceleration ε. We can do it using the equation

$$a_\tau = \varepsilon * r$$

$$\varepsilon = \frac{a_\tau}{r}$$

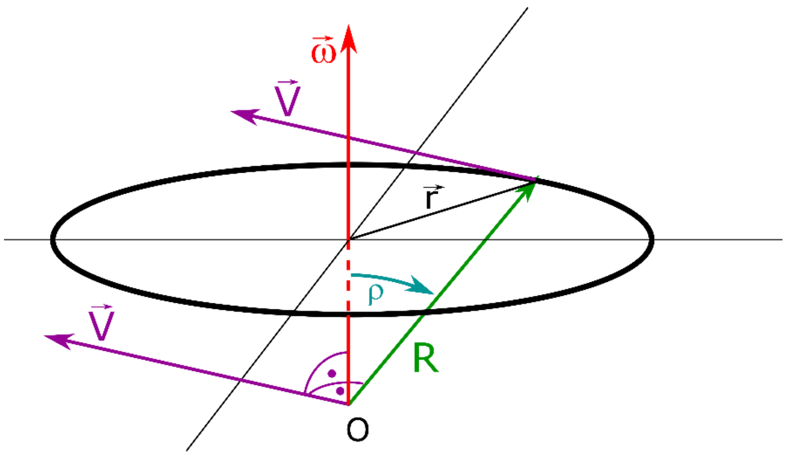
$$\varepsilon(t_1) = \frac{a_\tau}{r} = \frac{3}{1{,}5} = 2$$



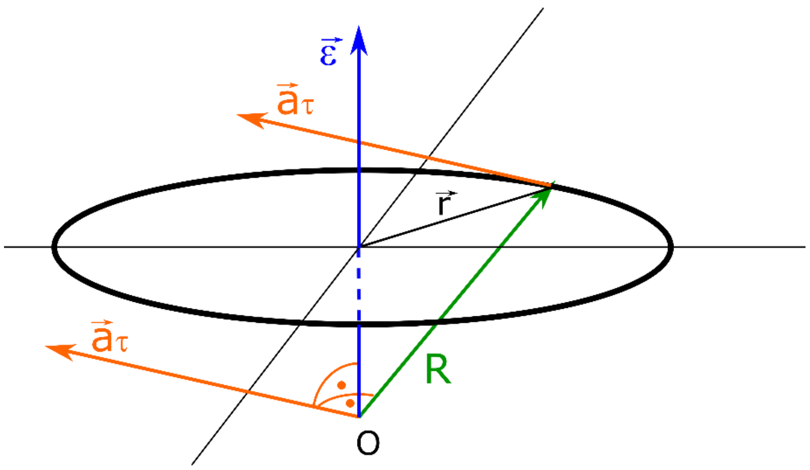
Since we have obtained a positive angular acceleration value, therefore, it will spin according to the rotation of the angular speed.

At the end, the relationships between vector quantities in the previous equations are presented.

The first figure is for linear and angular velocity



The second figure is related to the tangential acceleration



The last drawing is related to normal acceleration.